

Robust quantum state certification and uncertainty principles for total influence

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Abstract

We show that nonadaptive single-qubit Pauli measurements suffice to test whether an unknown n -qubit state ρ is ε -close to or $\mathcal{O}(\varepsilon)$ -far from an ideal target state $|\psi\rangle$, for all but a $2^{-\Omega(n)}$ fraction of target states. The test uses $\mathcal{O}(\varepsilon^{-2} \log(1/\delta))$ copies of ρ to achieve confidence $1 - \delta$, which is information-theoretically optimal even among protocols with arbitrary joint measurements.

The main technical innovation is an uncertainty principle for weighted generalizations of the total influence of Boolean functions. As a simple example, the unweighted variant states that $\mathbf{Inf}[f] + \mathbf{Inf}[\hat{f}] = \Omega(n)$, which is a natural hypercube analogue of the Heisenberg uncertainty principle (here $\hat{\cdot}$ denotes the $2^{-n/2}$ -normalized Fourier transform). The weighted case generalizes $\mathbf{Inf}[\cdot]$ and $\mathbf{Inf}[\hat{\cdot}]$ to Dirichlet energies associated with Glauber dynamics for certain dual measures on the cube.

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1 Introduction

A basic task in quantum information is *state certification*: testing whether a copy of an unknown state is close to a pure target state $|\psi\rangle$ (or far from it). This task has broad applications in quantum science, from benchmarking quantum devices to verifying the implementation of quantum algorithms [BOW19, ZH19, KR21, HPS25]. With no constraints on measurement capabilities, the optimal measurement is well-understood: it is simply the projective measurement $\{|\psi\rangle\langle\psi|, \mathbf{1} - |\psi\rangle\langle\psi|\}$. However, implementing this measurement is in general just as difficult as preparing $|\psi\rangle$ itself, and there is significant interest in finding state certification protocols that only require simple measurements—ideally of only a few qubits at a time. Remarkably, recent work has shown that single-qubit measurements suffice.

The first such result was by Huang, Preskill, and Soleimanifar [HPS25], who proposed a protocol consisting of non-adaptive single-qubit Pauli measurements, which can certify all but an exponentially-small fraction of n -qubit target states. In subsequent, independent works, Gupta, He, and O’Donnell [GHO25], and Li and Zhu [LZ25] proposed somewhat more intricate protocols consisting of single-qubit measurements that can certify *any* target state. [GHO25] also showed that adaptivity is *necessary* to get a single-qubit measurement protocol that can certify any state. In a more recent work, Abdul Sater et al. [SGM⁺25] describe a fully-efficient protocol (using only efficiently accessible information about the target state) to certify a useful, but restricted, class of states with Pauli measurements.

Unfortunately, all of the above protocols share a limitation, which is that they are non-robust: they can only certify whether the unknown state ρ is $\mathcal{O}(\varepsilon/n)$ -close or at least ε -far from the n -qubit target state $|\psi\rangle$. Said another way, these protocols only succeed in positively certifying lab states that are vanishingly close to the ideal state as the number of qubits n grows, making them difficult to apply in practice: if an experimentalist wishes to double their system size (*i.e.*, double n), the test will only remain usable if they simultaneously double their preparation accuracy. Moreover, one can show that this non-robustness is not just a limitation of the analysis, but is inherent to their protocols.¹

More recently, Coladangelo et al. [CLSW26] gave two protocols that make a substantial step towards removing this limitation. Their first protocol involves single-qubit measurements along with one $\log(n)$ -qubit measurement and achieves *constant* robustness: it can certify whether ρ is ε/C -close to or ε -far from $|\psi\rangle$, for a universal constant $C \approx 2$, independent of n . Their second protocol requires only single-qubit measurements, but only achieves $\log(n)$ robustness (*i.e.*, $C = \Theta(\log n)$).

The authors of [CLSW26] left open the tantalizing possibility that one could achieve the best of both worlds: a certification protocol for almost all target states that achieves *constant* robustness using only *single-qubit* measurements (non-adaptively). In particular, they conjectured [CLSW26, Conjecture 1] that a modification of their protocol (which amounts to running the [HPS25] test in two bases) should suffice. In this work we prove their conjecture in the affirmative. We show the following.

Theorem 1 (Constant robustness with single-qubit measurements). *There exists an efficient protocol using single-qubit measurements that, for all but a $2^{-\Omega(n)}$ fraction of pure n -qubit target states $|\psi\rangle$, satisfies the following. Given oracle access to a classical description of the target state $|\psi\rangle$ (via the model in Definition 1), and a single copy of an n -qubit mixed state ρ ,*

- (Completeness) *Outputs accept with probability at least $\langle\psi|\rho|\psi\rangle$.*
- (Soundness) *Outputs reject with probability at least $\frac{1-\langle\psi|\rho|\psi\rangle}{C} - o(1)$, for a universal constant $C > 1$ (*i.e.* approximately the infidelity up to a constant factor).*

Moreover, the single-qubit measurements are either all in the X -basis or all in the Z -basis, except for one final (non-Pauli) adaptive measurement.

¹See, for example, [CLSW26, Appendix A]

Reference	Largest measurement required	Copy complexity	Fraction of target states	Adaptivity	Robustness
[HPS25]	Single-qubit (Pauli)	$\mathcal{O}(n^2)$	$1 - 2^{-\Omega(n)}$	No	$\mathcal{O}(1/n)$
[GHO25]	Single-qubit (arbitrary)	$\mathcal{O}(n^2)$	All	Yes	$\mathcal{O}(1/n)$
[CLSW26]	Single-qubit (arbitrary)	$\mathcal{O}(\log^2 n)$	$1 - 2^{-\Omega(n)}$	Yes	$\mathcal{O}(1/\log n)$
[CLSW26]	$\log(n)$ -qubit (Clifford)	$\mathcal{O}(1)$	$1 - 2^{-\Omega(n)}$	No	$\Omega(1)$
This work	Single-qubit (Pauli)	$\mathcal{O}(1)$	$1 - 2^{-\Omega(n)}$	No	$\Omega(1)$

Figure 1: Comparing this work to previous protocols certifying all or almost all target states. Green cells denote the best possible for the given column note that it is impossible for all columns to be green at the same time by the adaptivity no-go theorem of [GHO25]. Accuracy ε and confidence δ are assumed to be constants in the quantities above. A t -robust protocol accepts states that are $t\varepsilon$ -close to the target states and rejects those which are ε -far. A concurrent update to a work of Du et al. [DLH⁺26], which we discuss below, also achieves the desiderata that we achieve in this work.

The completeness-soundness gap can be amplified via sequential repetition using more copies, and the small amount of adaptivity can be completely removed using classical shadow tomography. As a result, [Theorem 1](#) implies the following corollary: a copy-optimal test using exclusively non-adaptive single-qubit Pauli measurements, with arbitrarily low failure probability.

Corollary 1 (Optimal copy-complexity with with non-adaptive single-qubit Pauli measurements). *There exists an efficient protocol using only non-adaptive single-qubit Pauli measurements that, for all but a $2^{-\Omega(n)}$ fraction of pure n -qubit target states $|\psi\rangle$, satisfies the following. Given oracle access to a classical description of the target state $|\psi\rangle$ (via the model in [Definition 1](#)), parameters $\varepsilon, \delta > 0$, and $\mathcal{O}(\varepsilon^{-2} \log(1/\delta))$ copies of an arbitrary state ρ ,*

- (completeness) *Outputs accept with probability at least $1 - \delta$ if $\langle \psi | \rho | \psi \rangle \geq 1 - \frac{\varepsilon}{C}$.*
- (soundness) *Outputs reject with probability at least $1 - \delta$, if $\langle \psi | \rho | \psi \rangle \leq 1 - \varepsilon$.*

for some constant $C > 1$.

Moreover, the single-qubit measurements are either all in the X -basis or all in the Z -basis, except for one qubit that is (non-adaptively) measured in a random Pauli basis.

We pause here to make some comments on these results. Concretely, the results resolve Conjecture 1 of [CLSW26], and demonstrate that the two-basis test proposed in [CLSW26] yields optimal state certification guarantees (up to constants) for almost all states. Note that the restriction that the method works for *almost* all states is inherent (and not a limitation of the analysis) since the test is non-adaptive, and [GHO25] demonstrates that adaptivity is required to certify all states.

On a more technical level, our main conceptual contribution is a novel proof technique for analyzing the soundness of the two-basis test, which significantly departs from the previous proof strategy of [CLSW26]. In particular, we draw a novel connection between the soundness of this test and new uncertainty principles for total influence of Boolean functions and, especially, more general weighted versions of total influence. These uncertainty principles generalize the natural hypercube analog of the famous Heisenberg uncertainty principle, as we explain in [Section 2.3](#).

Concurrent work. Shortly before this work was posted, we were informed by the authors of [DLH⁺26] of an update to their paper, giving a test that also achieves constant robustness for almost all target states using Pauli measurements. We briefly elaborate on the differences between our

respective tests (for the unfamiliar reader, these differences should be more understandable after reading our technical overview).

In short, the test in [DLH⁺26] runs the [HPS25] test in a uniformly random Pauli basis without randomizing the location of the “leftover” qubit. Our test is even simpler (and its soundness was left as Conjecture 1 in [CLSW26]): it runs the [HPS25] test in one of only two bases, either all- Z or all- X , but randomizes the location of the leftover qubit. It is this two-basis picture that leads to the connections with uncertainty principles for total influences that we uncover here. As a result, the requirements on the oracle access to the target state needed for our test are in some sense lighter: we require queries to the amplitudes of 1-qubit states post-selected in only two bases, whereas the test of [DLH⁺26] requires an oracle that can answer such queries in 3^{n-1} potential Pauli bases.

The authors of [DLH⁺26] also make progress on one of the questions that we mention in Section 2.6, namely finding examples of natural families of states that are in the “good” set of target states: they show that random graph states, a subclass of stabilizer states, are in the “good” set.

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2 Technical Overview

2.1 The test

We first describe the (slightly adaptive) single-shot test that achieves the guarantees of Theorem 1. We will then describe the nonadaptive version that achieves the guarantee of Corollary 1.

Protocol 1 (One-shot (slightly adaptive) robust state certification).

Input: One copy of an n -qubit lab state ρ ; and access to the n -qubit target state $|\psi\rangle$ (via the access model described in Section 2.2).

1. Sample a measurement. Sample the following:

- A basis $B \in \{X, Z\}$ with equal probability,
- A holdout qubit $j \in [n]$ with uniform probability.

2. Measure the bulk of the lab state. Measure all the qubits $k \neq j$ in basis B , obtaining outcome string $b \in \{0, 1\}^{n-1}$ and 1-qubit post-measurement state $\rho_b^{(j,B)}$ on the holdout qubit j .

3. Compute target. Compute (a classical description of) $|\psi_b^{(j,B)}\rangle$, the 1-qubit state on the holdout qubit j resulting from measuring qubits $k \neq j$ of the target state $|\psi\rangle$ in basis B and obtaining outcome b . (The classical description of $|\psi_b^{(j,B)}\rangle$ is computed via the access model described in Section 2.2).

4. Measure the holdout qubit. Perform the projective measurement $\{|\psi_b^{(j,B)}\rangle\langle\psi_b^{(j,B)}|, \mathbf{1} - |\psi_b^{(j,B)}\rangle\langle\psi_b^{(j,B)}|\}$ on $\rho_b^{(j,B)}$. Accept if the first outcome is obtained.

What we show is that this protocol accepts with probability at least $\langle\psi|\rho|\psi\rangle$ (*i.e.* the fidelity

between ρ and ψ), and rejects with probability at least (approximately) $\frac{1 - \langle \psi | \rho | \psi \rangle}{C}$ for some constant $C > 1$ (i.e. the infidelity up to a constant factor). One can readily convert this one-shot protocol to one that distinguishes, with any desired success probability $1 - \delta$, a lab state ρ that has $1 - \frac{\varepsilon}{C'}$ fidelity with $|\psi\rangle$ from a ρ that has at most $1 - \varepsilon$ fidelity with $|\psi\rangle$ (for some constant $C' > 1$ closely related to C). The latter protocol can be obtained via a standard sequential repetition using $O(\varepsilon^{-2} \log(1/\delta))$ copies of ρ , which is information-theoretically optimal even among protocols that are allowed to make arbitrary entangled measurements.

The only source of adaptivity in the resulting protocol comes from Step 4 of the one-shot protocol, where the measurement basis depends on the previous outcome b . This adaptivity can be removed via classical shadow tomography, as it is done in [HPS25]. In short, the adaptive measurement on the holdout qubit j in Step 4 can be replaced by a non-adaptive measurement in a uniformly random choice of the X , Y , or Z basis. This allows one to compute (by classical post-processing) an unbiased estimator for the outcome of any adaptively-chosen measurement on the holdout qubit. This process is repeated $T = O(\varepsilon^{-2} \log(1/\delta))$ times (using that many copies of ρ), and the protocol accepts if the average of the T estimators is above the appropriate threshold. Note that the holdout qubit, as well as the post-measurement state, in general changes in each of the T repetitions, but this is fine: each of the T runs is an unbiased estimator of the one-shot acceptance probability (and since the runs are independent, we can invoke Chernoff). We refer to [HPS25] for more details about this application of classical shadows. The nonadaptive protocol is described below. The constant $C > 2$ in Protocol 2 is universal and independent of n . In our proofs, the constant we establish is only implicit. However according to numerical experiments, the tight constant appears to be $C < 5.5$.

Protocol 2 (Non-adaptive robust state certification).

Parameters: $\varepsilon, \delta > 0$.

Input: $T = O(\varepsilon^{-2} \log(1/\delta))$ copies of an n -qubit lab state ρ ; and access to the n -qubit target state $|\psi\rangle$ (via the access model described in Section 2.2).

1. Sample a measurement. Choose:

- A basis $B \in \{X, Z\}$ with equal probability,
- A holdout qubit $j \in [n]$ with uniform probability,
- A holdout basis $C \in \{X, Y, Z\}$ with uniform probability.

2. Measure the lab state. Measure the lab state ρ in the following single-qubit product basis: all qubits $k \neq j$ in basis B and the holdout qubit j in basis C . This yields an outcome bit $c \in \{0, 1\}$ for the holdout qubit and an outcome string $b \in \{0, 1\}^{n-1}$ for the rest of the qubits.

3. Classical postprocessing. Let $\rho_b^{(j,B)}$ and $|\psi_b^{(j,B)}\rangle$ be as in Protocol 1. Compute the classical shadow tomography estimate ω for

$$\langle \psi_b^{(j,B)} | \rho_b^{(j,B)} | \psi_b^{(j,B)} \rangle$$

corresponding to basis C and outcome c .

4. Repeat. Repeat steps 1–3 T times to obtain estimates $\omega_1, \dots, \omega_T$. Let $\hat{\omega} = \frac{1}{T} \sum_{i=1}^T \omega_i$. Accept if $\hat{\omega} \geq 1 - \frac{3\varepsilon}{2C}$.

The protocol above accepts with probability at least $1 - \delta$ if $\langle \psi | \rho | \psi \rangle > 1 - \frac{\varepsilon}{C}$, and rejects with probability at least $1 - \delta$ if $\langle \psi | \rho | \psi \rangle < 1 - \varepsilon$. The latter follows in a straightforward way from the guarantees of Protocol 1 and the fact that classical shadow tomography gives an unbiased estimator for the true overlap.

2.2 Access model

As in earlier protocols [HPS25, GHO25] (henceforth HPS and GHO) we assume classical access to an oracle-based description of the target state $|\psi\rangle$. The access model that we work with is, in terms of strength, between that of HPS and GHO. In particular, we require knowledge of amplitudes of the target state in *two* bases: standard and Hadamard. For comparison, HPS only requires amplitudes in the standard basis, whereas GHO require amplitudes in all product bases (but is capable of certifying *all* target states, rather than “almost all”).

We acknowledge that our access model creates a bit of a tension: Haar-random states are highly entangled and exponentially complex, whereas states for which this kind of amplitude access can be simulated efficiently are generally more structured and tractable. For example, tensor network states with polynomial bond dimension allow efficient amplitude queries but are low-entangled, and neural quantum states typically support efficient queries in only one basis. This means that our test will generally require inefficient classical post-processing. We note that, while GHO require an even stronger access model, their certification works for all states (which thus includes classes of states for which the access model is efficiently reproducible). That said, we find the existence of a robust test that uses few-qubit measurements surprising even from a purely information-theoretic standpoint.

Definition 1. *We consider an oracle model which accepts queries in the format $(b, \mathbf{B})$ for $b \in \{0, 1\}^n$ and $\mathbf{B} \in \{\text{std}, \text{had}\}$, and returns the complex number*

$$O_\psi(b, \mathbf{B}) = \begin{cases} \langle b | \psi \rangle & \text{if } \mathbf{B} = \text{std} \\ \langle b | H^{\otimes n} | \psi \rangle & \text{if } \mathbf{B} = \text{had} . \end{cases}$$

This access model allows us to explicitly and efficiently learn the leftover state on the last qubit, conditioned on the outcome of any measurement on $|\psi\rangle$ used in our protocols.

2.3 Uncertainty principles for (weighted) total influence

The correctness of our protocol relies centrally on a new uncertainty principle for a certain weighted generalization of the total influence of Boolean functions. This uncertainty principle may be of independent interest, so we describe it here in self-contained form. The connection to our protocols is explained in the next subsection, [Section 2.4](#).

Before stating the full theorem, let us record a simple unweighted version of the phenomenon. For any function $h : \{0, 1\}^n \rightarrow \mathbb{C}$, define its total influence via

$$\mathbf{Inf}[h] = \sum_{i,x} \left| \frac{h(x) - h(x+i)}{2} \right|^2 ,$$

where here and throughout, for an index $i \in [n]$, we use the shorthand notation $x+i$ to denote the string where we flip the i -th bit of x . Then we have the following.

Proposition 1 (Uncertainty principle for unweighted total influence). *For any $h : \{0, 1\}^n \rightarrow \mathbb{C}$ with $\mathbf{E}_x |h(x)|^2 = 1$,*

$$\mathbf{Inf}[h] + \mathbf{Inf}[\widehat{h}] \gtrsim n .$$

Here and throughout, for quantities A and B , the notation $A \gtrsim B$ means that there exists a universal constant $c > 0$ such that $A \geq cB$. Also, the Fourier transform $\widehat{\cdot}$ will throughout bear the unitary normalization $2^{n/2}$. That is, for $s \in \{0, 1\}^n$ we take the convention

$$\widehat{f}(s) = 2^{-n/2} \sum_{x \in \{0,1\}^n} f(x) (-1)^{\sum_i x_i s_i} .$$

The total influence, **Inf**, is a central quantity in the analysis of Boolean functions and plays an important role throughout theoretical computer science [O'D14]. Despite this, an uncertainty principle for **Inf** does not appear to be well-known within the TCS community. The proof of [Proposition 1](#) is simple however, and can be obtained for example by specializing a graph-theoretic uncertainty principle due to Benedetto and Koprowski [BK15] to the hypercube graph. For completeness, we give a direct proof in [Section 2.5.1](#). As elaborated in [BK15], this influence uncertainty principle is the natural analogue to the additive Heisenberg uncertainty principle.

The correctness of our test reduces to the following *weighted* generalization of the influence-uncertainty phenomenon. For any probability measure μ on $\{0, 1\}^n$ and any $h : \{0, 1\}^n \rightarrow \mathbb{C}$, define the μ -weighted total influence to be

$$\mathbf{Inf}_\mu[h] = \sum_{i,x} \frac{2\mu(x)\mu(x+i)}{\mu(x) + \mu(x+i)} \left| \frac{h(x) - h(x+i)}{2} \right|^2.$$

With this definition, we have the following uncertainty principle for $\mathbf{Inf}_\mu$.

Theorem 2 (Uncertainty principle for weighted total influence). *With probability $1 - 2^{-\Omega(n)}$ over a Haar-random $f : \{0, 1\}^n \rightarrow \mathbb{C}$ with $\|f\|_2 = 1$, for all $g : \{0, 1\}^n \rightarrow \mathbb{C}$ with $\|g\|_2 = 1$ and $\langle f, g \rangle = 0$,*

$$\mathbf{Inf}_{\mu_f} \left[\frac{g}{f} \right] + \mathbf{Inf}_{\mu_{\hat{f}}} \left[\frac{\hat{g}}{\hat{f}} \right] \gtrsim n.$$

Here μ_f (resp. $\mu_{\hat{f}}$) is defined by $\mu_f(x) = |f(x)|^2$ (resp. $\mu_{\hat{f}}(s) = |\hat{f}(s)|^2$).

It is worth mentioning that $\mathbf{Inf}_\mu$ is precisely the Dirichlet energy associated with continuous-time Glauber dynamics for the stationary distribution μ . See, for example, [LP17, §3.3.2 and §13.2.1]. In comparison to the unweighted version, the uncertainty principle of [Theorem 2](#) seems to demand a dramatically more involved proof, spanning [Sections 5–8](#).

2.4 Proof ideas I: From state certification to uncertainty principles

As mentioned earlier, the crux is to analyze the [Protocol 1](#), *i.e.* the one-shot protocol. This will be our focus for the rest of the technical overview.

Notice that the acceptance probability in [Protocol 1](#) can be written as follows:

$$\Pr[\text{accept}] = \frac{1}{2} \left(\mathbf{E}_{\substack{j \sim [n] \\ x \sim \text{Meas}(\rho, X, [n] \setminus j)}} \langle \psi_x^{(j,X)} | \rho_x^{(j,X)} | \psi_x^{(j,X)} \rangle + \mathbf{E}_{\substack{j \sim [n] \\ z \sim \text{Meas}(\rho, Z, [n] \setminus j)}} \langle \psi_z^{(j,Z)} | \rho_z^{(j,Z)} | \psi_z^{(j,Z)} \rangle \right),$$

where $|\psi_b^{(j,B)}\rangle$ and $\rho_b^{(j,B)}$ for $B \in \{X, Z\}$ and $b \in \{0, 1\}^{n-1}$ are defined as in [Protocol 1](#): they are the post-measurement state of the j -th qubit when measuring the rest of the qubits of, respectively, $|\psi\rangle$ and ρ in basis B and obtaining outcome b . And $\text{Meas}(\rho, B, [n] \setminus j)$ is the corresponding distribution of over the outcome b (when measuring ρ).

It is straightforward to see that $\Pr[\text{accept}] \geq \langle \psi | \rho | \psi \rangle$: we can equivalently write $\Pr[\text{accept}] = \text{Tr}[A\rho]$ for an appropriate positive-semidefinite operator A , which has $|\psi\rangle$ as a $+1$ eigenvector (see [Section 4](#) for the definition of A). Then, it follows that $\text{Tr}[A\rho] \geq \langle \psi | \rho | \psi \rangle$. The bulk of the work is to prove an *upper bound* on $\Pr[\text{accept}]$ in terms of fidelity. Ultimately, we will show the following.

Theorem 3. *With all but $2^{-\Omega(n)}$ probability over a Haar-random target state $|\psi\rangle$, for all lab states ρ ,*

$$\mathbf{E}_{\substack{j \sim [n] \\ x \sim \text{Meas}(\rho, X, [n] \setminus j)}} \langle \psi_x^{(j,X)} | \rho_x^{(j,X)} | \psi_x^{(j,X)} \rangle + \mathbf{E}_{\substack{j \sim [n] \\ z \sim \text{Meas}(\rho, Z, [n] \setminus j)}} \langle \psi_z^{(j,Z)} | \rho_z^{(j,Z)} | \psi_z^{(j,Z)} \rangle \leq 2 - c(1 - \langle \psi | \rho | \psi \rangle)$$

Here $c > 0$ is a universal constant.

Accounting for the factor of $\frac{1}{2}$ in the test, we arrive at

$$\langle \psi | \phi | \psi \rangle \leq \Pr[\text{accept}] \leq 1 - \frac{c}{2} (1 - \langle \psi | \rho | \psi \rangle).$$

Reducing to a formulation on the hypercube. By a straightforward argument, it suffices to argue the case where $\rho = |\phi\rangle\langle\phi|$ is pure and orthogonal to $|\psi\rangle$ (the case of general ρ can then be understood by writing ρ in terms of the eigenbasis of the “acceptance” operator A mentioned earlier, which includes $|\psi\rangle$ as a top eigenvector). Our main technical result is the following.

Proposition 2 (Orthogonal pure state version). *With probability $1 - 2^{-\Omega(n)}$ over a Haar-random target state $|\psi\rangle$, for all pure states $|\phi\rangle$ orthogonal to $|\psi\rangle$,*

$$\mathbf{E}_{\substack{j \sim [n] \\ x \sim \text{Meas}(|\phi\rangle, X, [n] \setminus j)}} |\langle \psi_x^{(j,X)} | \phi_x^{(j,X)} \rangle|^2 + \mathbf{E}_{\substack{j \sim [n] \\ z \sim \text{Meas}(|\phi\rangle, Z, [n] \setminus j)}} |\langle \psi_z^{(j,Z)} | \phi_z^{(j,Z)} \rangle|^2 \leq 2 - c,$$

for a universal constant $c > 0$ independent from n .

Let us define the functions $f, g : \{0, 1\}^n \rightarrow \mathbb{C}$ via

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} f(x) |x\rangle \quad \text{and} \quad |\phi\rangle = \sum_{x \in \{0,1\}^n} g(x) |x\rangle.$$

For convenience, we will also switch to the natural notation $|f\rangle = |\psi\rangle$, and $|g\rangle = |\phi\rangle$.

To formulate [Proposition 2](#) in terms of f and g (and their Fourier transforms), we can define the “edgewise averages”

$$\begin{aligned} \mathbf{Avg}_f[g] &:= \frac{1}{2n} \sum_i \sum_x \frac{\left| \overline{f(x)}g(x) + \overline{f(x+i)}g(x+i) \right|^2}{|f(x)|^2 + |f(x+i)|^2} \in [0, 1], \\ \mathbf{Avg}_{\widehat{f}}[\widehat{g}] &:= \frac{1}{2n} \sum_i \sum_s \frac{\left| \overline{\widehat{f}(s)}\widehat{g}(s) + \overline{\widehat{f}(s+i)}\widehat{g}(s+i) \right|^2}{|\widehat{f}(s)|^2 + |\widehat{f}(s+i)|^2} \in [0, 1], \end{aligned}$$

noting that

$$\mathbf{Avg}_f[g] = \mathbf{E}_{\substack{j \sim [n] \\ x \sim \text{Meas}(|\phi\rangle, X, [n] \setminus j)}} |\langle \psi_x^{(j,X)} | \phi_x^{(j,X)} \rangle|^2 \quad \text{and} \quad \mathbf{Avg}_{\widehat{f}}[\widehat{g}] = \mathbf{E}_{\substack{j \sim [n] \\ z \sim \text{Meas}(|\phi\rangle, Z, [n] \setminus j)}} |\langle \psi_z^{(j,Z)} | \phi_z^{(j,Z)} \rangle|^2.$$

(The factor of $\frac{1}{2}$ in $\mathbf{Avg}_f[g]$ and $\mathbf{Avg}_{\widehat{f}}[\widehat{g}]$ is because the sum double counts each inner product.) With these definitions, [Proposition 2](#) is equivalent to the following “uncertainty principle”, which, as we will show along the way, is equivalent to the uncertainty principle for weighted influences of [Theorem 2](#).

Theorem 4. *With probability $1 - 2^{-\Omega(n)}$ over a Haar-random $f : \{0, 1\}^n \rightarrow \mathbb{C}$, $\|f\|_2 = 1$, for all $g : \{0, 1\}^n \rightarrow \mathbb{C}$, $\|g\|_2 = 1$ with $\langle f, g \rangle = 0$,*

$$\mathbf{Avg}_f[g] + \mathbf{Avg}_{\widehat{f}}[\widehat{g}] \leq 2 - c,$$

where $c > 0$ is a universal constant independent of n .

Before explaining our approach and relating this inequality to other types of uncertainty, we pause to note one final formulation that will be used several times. The averages can be expressed as quadratic forms in g , namely as

$$\mathbf{Avg}_f[g] =: \langle g, M_f g \rangle$$

for an appropriate matrix parameterized by f (this is the matrix corresponding to the “acceptance” operator A mentioned earlier). Similarly,

$$\mathbf{Avg}_{\widehat{f}}[\widehat{g}] = \langle \widehat{g}, M_{\widehat{f}} \widehat{g} \rangle = \langle g | H^{\otimes n} M_{\widehat{f}} H^{\otimes n} | g \rangle =: \langle g, \widehat{M_{\widehat{f}}} g \rangle.$$

The uncertainty principle thus has the following spectral interpretation: individually, the two operators M_f and $\widehat{M_{\widehat{f}}}$ have an inverse-polynomial spectral gap:

$$\text{Spec}(M_f) = \{1, 1 - \Theta(\frac{1}{n}), \dots\} \quad \text{and} \quad \text{Spec}(\widehat{M_{\widehat{f}}}) = \{1, 1 - \Theta(\frac{1}{n}), \dots\},$$

where the top eigenspace is in both cases one-dimensional with eigenvector $|f\rangle$. The latter follows directly from the analysis of the “one-basis” test of [HPS25] (and the fact that this analysis is tight can be seen, for instance, via an example described in [CLSW26, Appendix A]). Yet, what we show is that

$$\text{Spec}(M_f + \widehat{M_{\widehat{f}}}) = \{2, 2 - c, \dots\},$$

for a constant $c > 0$ independent of n .

As we present the proof ideas in the rest of the technical overview, we will arrive along the way at a reformulation of this uncertainty principle in terms of the total influences (as stated earlier in Theorem 2).

2.5 Proof ideas II: Proving the uncertainty principles

We explain our proof approach in three stages: an “unweighted” toy case, the “phase state” case, and the general case.

2.5.1 Unweighted toy case: an uncertainty principle for total influence

Consider the uniform unweighted case, where we replace each occurrence of $f(x)$ or $\widehat{f}(s)$ with $2^{-n/2}$ (note that this is a toy case because such a function f does not exist!). Then, instantiating the previous definition of $\mathbf{Avg}_f[g]$ with the constant function $f = 2^{-n/2}$ we have

$$\mathbf{Avg}[g] = \frac{1}{4n} \sum_i \sum_x |g(x) + g(x+i)|^2.$$

A simple, but revealing, observation is that, since $\|g\|_2 = 1$, we can rewrite $\mathbf{Avg}[g]$ as

$$\mathbf{Avg}[g] = 1 - \frac{1}{n} \sum_{i,x} \left(\frac{g(x) - g(x+i)}{2} \right)^2 = 1 - \frac{1}{n} \mathbf{Inf}_{\text{count}}[g], \quad (1)$$

where $\mathbf{Inf}_{\text{count}} = \mathbf{Inf}[2^{n/2} \cdot g]$ is the *total influence* of g under the counting measure (which is the correct scaling for g). The identity (1) can be straightforwardly checked; for the proof of a much more general identity, we refer the reader to Appendix C.

In this toy picture, the desired uncertainty principle for $\mathbf{Avg}$ is then, for a universal $c > 0$,

$$\mathbf{Avg}[g] + \mathbf{Avg}[\widehat{g}] \leq 2 - c \iff \mathbf{Inf}_{\text{count}}[g] + \mathbf{Inf}_{\text{count}}[\widehat{g}] \geq cn.$$

It turns out this uncertainty principle is true and for example follows from a graph-theoretic uncertainty principle of [BK15] specialized to the hypercube. We give a short, self-contained proof next.

Proposition 3 (Uncertainty principle for total influence). *Let $g : \{0, 1\}^n \rightarrow \mathbb{C}$ with $\|g\|_2 = 1$. Then*

$$\mathbf{Inf}_{\text{count}}[g] + \mathbf{Inf}_{\text{count}}[\widehat{g}] \geq \left(1 - \frac{1}{\sqrt{2}}\right)n \quad (2)$$

Moreover, this inequality is sharp.

Note that it is a standard fact that

$$\mathbf{Inf}_{\text{count}}[g] = \sum_s |s| \cdot |\widehat{g}(s)|^2, \quad (3)$$

and thus [Proposition 3](#) is equivalent to

$$\sum_s |s| \cdot |\widehat{g}(s)|^2 + \sum_x |x| \cdot |g(x)|^2 \geq \left(1 - \frac{1}{\sqrt{2}}\right)n. \quad (4)$$

Proof of [Proposition 3](#). Let $D = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Inspecting (4), one observes that

$$\begin{aligned} \mathbf{Inf}_{\text{count}}[g] &= \langle g | H^{\otimes n} \left(\sum_{j=1}^n I^{\otimes j-1} \otimes D \otimes I^{n-j} \right) H^{\otimes n} | g \rangle \\ \text{and} \quad \mathbf{Inf}_{\text{count}}[\widehat{g}] &= \langle g | \sum_{j=1}^n I^{\otimes j-1} \otimes D \otimes I^{n-j} | g \rangle, \end{aligned}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} / \sqrt{2}$ is the 2×2 Hadamard matrix. The best constant in (2) is therefore given by the minimum eigenvalue of the operator

$$A := \sum_{j=1}^n I^{\otimes j-1} \otimes \underbrace{(HDH + D)}_{:=M} \otimes I^{\otimes n-j} = \sum_{j=1}^n I^{j-1} \otimes M \otimes I^{n-j}.$$

Let $\{|u\rangle, |v\rangle\}$ be an eigenbasis for M and note that the $(I^{j-1} \otimes M \otimes I^{n-j})$'s are simultaneously diagonalized by the eigenbasis $\{|u\rangle, |v\rangle\}^{\otimes n}$. As a result, the spectrum of A comprises n -term sums of elements of $\text{Spec}(M)$, and in particular

$$\lambda_{\min}(A) = n\lambda_{\min}(M).$$

The last quantity $\lambda_{\min}(M)$ is determined by direct calculation. $\square$

We remark that this influence uncertainty principle, especially in its formulation in (4), is in some sense the right hypercube analogue of the Heisenberg uncertainty principle. This is elaborated in [\[BK15\]](#).

2.5.2 The phase state case and some key ideas

Let us now return to the f -weighted quantities $\mathbf{Avg}_f[\cdot]$ and $\mathbf{Avg}_{\widehat{f}}[\cdot]$, where recall that the goal is to show

$$\mathbf{Avg}_f[g] + \mathbf{Avg}_{\widehat{f}}[\widehat{g}] \leq 2 - c,$$

for some universal constant $c > 0$. In this subsection, we will focus on the crucial special case of f a *phase state*, i.e., $|f(x)|^2 = 2^{-n}$ for all x . Note that, while in the toy case the uncertainty principle holds for all g with $\|g\|_2 = 1$, once we move to the f -weighted case, we must restrict our attention to g that are orthogonal to f , as in the hypothesis of [Theorem 4](#) (and this restriction is necessary since, for example, $\mathbf{Avg}_f[f] + \mathbf{Avg}_{\widehat{f}}[\widehat{f}] = 2$). Since we seek a theorem holding w.h.p. over a random f , it is natural to attempt to prove concentration of the operator norm of $M_f + \widehat{M}_{\widehat{f}}$ when restricted to the subspace of allowed g 's. Unfortunately, the set of valid g 's is now a “moving target” (moving with f), and this seems to frustrate naive spectral approaches, such as those used in the toy unweighted case.

The first key idea in this work is a simple transformation that both decouples f from g and recovers a connection to total influence. Specifically, we factor out a copy of f from g , which is

possible with probability 1 over Haar-random f and always possible for f 's coming from phase states (the latter being our current focus). Formally, define $q : \{0, 1\}^n \rightarrow \mathbb{C}$ via $g = fq$, and note that

$$\|g\|_2^2 = 1 \iff \sum_x |f(x)|^2 |q(x)|^2 = \|q\|_{L^2(\text{unif})}^2 = 1,$$

and

$$\langle f, g \rangle = 0 \iff \sum_x |f(x)|^2 q(x) = \mathbf{E}_{x \sim \text{unif}} q = 0,$$

where we have used that $|f(x)|^2 = 2^{-n}$ for all x . Recall that in the formulation of the uncertainty principle from [Theorem 4](#), we care precisely about functions g satisfying the two conditions on the LHS above. Thus the class of g “compatible” with f is exactly those $g = fq$ where q is nothing but a balanced Boolean function, with standard L^2 normalization with respect to the uniform probability measure on $\{0, 1\}^n$.

Moreover, with this definition of q ,

$$\begin{aligned} \mathbf{Avg}_f[g] &= \frac{1}{2n} \sum_i \sum_x \frac{||f(x)|^2 q(x) + |f(x+i)|^2 q(x+i)|^2}{|f(x)|^2 + |f(x+i)|^2} \\ &= \frac{1}{4n} \sum_i \mathbf{E}_{x \sim \text{unif}} |q(x) + q(x+i)|^2 \\ &= 1 - \frac{1}{n} \sum_i \mathbf{E}_{x \sim \text{unif}} \left(\frac{|q(x) - q(x+i)|}{2} \right)^2 \\ &= 1 - \frac{1}{n} \mathbf{Inf}[q], \end{aligned} \tag{5}$$

where $\mathbf{Inf}$ is the standard total influence. So we have conveniently recovered, for the standard-basis side quantity $\mathbf{Avg}_f[g]$, an analogous expression to the earlier [Equation \(1\)](#), up to the correct normalization for the influence, and the passage from g to q .

One may then hope the rest of the story matches, and we recover $\mathbf{Inf}[\widehat{q}]$ on the Fourier side. Unfortunately this is not what happens; making this same substitution into $\mathbf{Avg}_{\widehat{f}}[\widehat{g}]$ does *not* yield $\mathbf{Inf}[\widehat{q}]$ but the following expression:

$$\mathbf{Avg}_{\widehat{f}}[\widehat{fq}] = \frac{1}{2n} \sum_i \sum_s \frac{|\widehat{f}(s)\widehat{fq}(s) + \widehat{f}(s+i)\widehat{fq}(s+i)|^2}{|\widehat{f}(s)|^2 + |\widehat{f}(s+i)|^2}. \tag{6}$$

In dual fashion to the standard basis side, this can be simplified to $1 - \frac{1}{n} \mathbf{Inf}[\widehat{g}/\widehat{f}]$, but $\widehat{g}/\widehat{f}$ has no simple algebraic relationship to $\widehat{q}$, and we thus cannot pass to $1 - \frac{1}{n} \mathbf{Inf}[\widehat{q}]$.

Despite this, it turns out that one can (with significant effort) show the following concentration inequality: for essentially² any fixed q , we have

$$\Pr_f \left[\mathbf{Avg}_{\widehat{f}}[\widehat{fq}] \geq \frac{1}{2} + t \right] \leq \exp(-ct^6 2^{n/3}). \tag{7}$$

Note this observation is not immediately useful because the quantifiers are in the wrong order: recall that we want a statement asserting that with high probability over f , it holds that for all q , $\mathbf{Avg}_{\widehat{f}}[\widehat{fq}]$ is well-behaved, whereas in (7) we are fixing a q before selecting a random f . But we *could* hope to correct the quantifier order via a union bound over an ε -net for the q 's, provided this net is not too large.

²We argue this concentration for all q satisfying certain technical regularity conditions. This suffices for the argument that follows.

What set of q 's should we use? The covering number of the set of *all* q 's is unfortunately too large: $\exp(\Omega(2^n))$ for $\varepsilon = \mathcal{O}(1)$. But recall that the standard-basis quantity $\mathbf{Avg}_f[g]$ is directly related to the total influence of q —and it is well-known that the set of *low-influence* q 's has small covering number. Connecting influence to the concentration phenomenon in (7) is the second key idea of this work, and we do so via a win-win argument. In the first case, $\mathbf{Inf}[q]$ is large, which means $\mathbf{Avg}_f[g]$ is small and we are already done. In the second case, $\mathbf{Inf}[q]$ is small and the ε -net argument succeeds in switching quantifiers in (7). In more detail, we argue as follows. Fix a sufficiently small $\kappa > 0$.

1. Let f be a random phase function. Let g be compatible with f (i.e. $\|g\| = 1$ and $\langle f, g \rangle = 0$), and let $q = g/f$.
2. If $\mathbf{Inf}[q] > \kappa n$ for some $\kappa > 0$, then, by (5), $\mathbf{Avg}_f[g] < 1 - \kappa$ and we are done: $\mathbf{Avg}_f[g] + \mathbf{Avg}_{\widehat{f}}[\widehat{g}] < 2 - \kappa$.
3. Otherwise, $\mathbf{Inf}[q] \leq \kappa n$.
 - (a) Since $\mathbf{Inf}[q] = \sum_s |s| \cdot |\widehat{q}(s)|^2$, small influence forces q to be approximately a *low-degree* polynomial. By a straightforward argument, there is a 0.01-net (with respect to L^2 distance) $Q_\kappa^{(0.01)}$ of such q with cardinality $|Q_\kappa^{(0.01)}| \leq \exp(c \cdot 2^{\mathcal{O}(\sqrt{\kappa}n)})$.
 - (b) For κ sufficiently small, the concentration in (7) beats the 0.01-net cardinality $|Q_\kappa^{(0.01)}|$. Thus by a union bound, w.h.p. over f , all $q^* \in Q_\kappa^{(0.01)}$ have $\mathbf{Avg}_{\widehat{f}}[\widehat{f}q^*] \leq 0.51$. Moreover, $\mathbf{Avg}_{\widehat{f}}[\widehat{f}\cdot]$ can easily be seen to be 1-Lipschitz, so we conclude that w.h.p. over f ,

$$\mathbf{Avg}_{\widehat{f}}[\widehat{f}q] \leq 0.52 \text{ for all } q \text{ with } \mathbf{Inf}[q] \leq \kappa n.$$

Thus in this case, w.h.p. over f , $\mathbf{Avg}_f[g] + \mathbf{Avg}_{\widehat{f}}[\widehat{g}] \leq 1.52$.

4. Putting things together, we conclude that w.h.p. over f , $\mathbf{Avg}_f[g] + \mathbf{Avg}_{\widehat{f}}[\widehat{g}] \leq \max\{2 - \kappa, 1.52\}$.

Although this argument is not strictly necessary for our general f result, we include a formal version of it as an interlude between Sections 5 and 6. For the case of general f to be outlined shortly in Section 2.5.3, several more obstacles will be lurking, but the general shape of the argument is shared.

Concentration bounds for $\mathbf{Avg}_{\widehat{f}}[\widehat{f}q]$: Proof of Equation (7). We now briefly describe the ideas behind the proof of the concentration bound in (7):

$$\Pr_f \left[\mathbf{Avg}_{\widehat{f}}[\widehat{f}q] \geq \frac{1}{2} + t \right] \leq \exp(-ct^6 2^{n/3}).$$

The main technical challenge is that (for a fixed q) while the numerator of $\mathbf{Avg}_{\widehat{f}}[\widehat{f}q]$, as in Equation (6), is a quartic polynomial in the entries of f —and thus amenable to concentration of polynomial chaos arguments—the presence of the denominators seems to frustrate all off-the-shelf concentration approaches. Moreover, the exponentially many terms in the sum are all mutually dependent (albeit weakly), and naive bounded difference (Efron–Stein) and Herbst-type arguments also do not work because $\mathbf{Avg}_{\widehat{f}}[\widehat{f}q]$ is highly non-differentiable in the $f(x)$'s.

Ultimately we overcome this barrier as follows: we smooth each summand of Equation (7) by adding a carefully chosen term $\tau > 0$ to the denominator and separately control both the smoothed version $\mathbf{Avg}_{\widehat{f}}^{(\tau)}[\widehat{f}q]$ and the error term $E = |\mathbf{Avg}_{\widehat{f}}^{(\tau)}[\widehat{f}q] - \mathbf{Avg}_{\widehat{f}}[\widehat{f}q]|$. This fixes the differentiability of the average, but seems to simply move the problem into the E term. However, we show that E is upper-bounded by another quantity that *is* differentiable and indeed concentrates very strongly around 0. This general perspective of smoothing and then differentially bounding the smoothing error will also be crucially utilized in the Haar random setting that we discuss shortly.

Overall, we have completed the description of the main ideas that go into proving [Theorem 4](#) for the restricted class of f corresponding to phase states. Restated as an uncertainty principle for total influences, we have that, w.h.p. over phase functions f , $\|f\|_2 = 1$, for all $g : \{0, 1\}^n \rightarrow \mathbb{C}$, $\|g\|_2 = 1$ with $\langle f, g \rangle = 0$,

$$\mathbf{Inf} \left[\frac{g}{f} \right] + \mathbf{Inf} \left[\frac{\widehat{g}}{\widehat{f}} \right] \geq C \cdot n, \quad (8)$$

for a universal constant $C > 0$.

Looking ahead to the full Haar-random f case, we will only be able to use the above win-win argument on a “good part” of the hypercube where $|f(x)|^2$ is sufficiently close to its expectation (notice that in the phase state case we do not have to worry about this!), and we will have to handle the rest of the functions by other means. We explain this in the next subsection.

2.5.3 The full monty: Haar-random states and the good-bad partition of $\{0, 1\}^n$

We now turn our attention to the case of a Haar random f . Let us see what happens when we factor an f out of g for general f (not necessarily with $|f|^2 \equiv 2^{-n}$). Similarly to the above, we have

$$\|g\|_2^2 = 1 \iff \sum_x |f(x)|^2 |q(x)|^2 = 1$$

and

$$\langle f, g \rangle = 0 \iff \sum_x |f(x)|^2 q(x) = 0.$$

This suggests that for a given f we should define the probability measure $\mu_f(x) = |f(x)|^2$, whereupon the compatibility requirements for q can be written as

$$\|q\|_{L^2(\mu_f)}^2 = 1 \quad \text{and} \quad \mathbf{E}_{x \sim \mu_f} q(x) = 0.$$

Crucially, we see that in the q -world, compatibility is governed only by the moduli of the $f(x)$ ’s, not their arguments. Thus from now on we will say that a q is μ_f -compatible if it satisfies the two equations above.

This phenomenon carries through to $\mathbf{Avg}_f$. When viewed as a function of q , it only depends on μ_f , and a straightforward calculation gives:

$$\begin{aligned} \mathbf{Avg}_f[g] &= 1 - \frac{1}{2n} \sum_i \sum_x \frac{|f(x)|^2 |f(x+i)|^2}{|f(x)|^2 + |f(x+i)|^2} \left| \frac{g(x)}{f(x)} - \frac{g(x+i)}{f(x+i)} \right|^2 \\ &= 1 - \frac{1}{n} \sum_{i,x} \frac{2\mu_f(x)\mu_f(x+i)}{\mu_f(x) + \mu_f(x+i)} \left| \frac{q(x) - q(x+i)}{2} \right|^2 \\ &=: 1 - \frac{1}{n} \mathbf{Inf}_{\mu_f}[q], \end{aligned}$$

where we have defined $\mathbf{Inf}_{\mu_f}[q]$ to match precisely the latter expression. Thus, $\mathbf{Avg}_f[g]$ is linearly related to a weighted version of the influence of q , where each edge is weighted by the harmonic mean of μ_f at each incident vertex. What’s more, since f is Haar-random, by standard concentration inequalities, μ_f is approximately flat: $\mu_f(x) \approx 2^{-n}$ for most x (if this approximation held uniformly for all x , then we would more or less be back in the phase state setting which we already understand).

Given the apparent dependence on the measure induced by f , rather than f itself, we are naturally led to the following approach. First, stratify the f ’s by their measure; *i.e.*, define

$$F_\mu = \{f : \mu_f = \mu\}.$$

Then, for each “typical” μ , repeat the win-win argument from the phase-state case, using the randomness in $f \sim_{\text{unif}} F_\mu$. More concretely, for any μ we have two cases: first, if $\mathbf{Inf}_\mu[q] > \kappa n$ for a suitable constant κ , then we are done; second, if $\mathbf{Inf}_\mu[q] \leq \kappa n$, it is reasonable to try repeating the ε -net argument from before. And indeed, the fixed- q and random- f concentration bound (7) holds in this adjusted setting where f is drawn uniformly from F_μ .

However, a problem appears when we try to take the union bound in the ε -net argument. It turns out that, unlike in the phase state case, a bound on $\mathbf{Inf}_\mu[q]$ does not, for typical μ , meaningfully control the covering number of μ -compatible q ’s. Roughly, the reason is that $\mathbf{Inf}_\mu[q]$ is not sensitive to changes in q ’s value at the points x with both: (i) abnormally large mass $\mu(x)$ and (ii) neighbors that are “normal-sized.” There are many such points in a typical μ , and this allows one to “hide” a large ε -net by varying the values of q at such x . To be clear, this is not a weakness in our analysis of the covering number; one really can show the existence of a $\Omega_\kappa(N)$ -dimensional subspace of functions q , all of which have $\mathbf{Inf}_\mu[q] \leq \kappa n$ —see [Appendix B](#).

Our strategy for circumventing this barrier is to partition the domain into two sets: $A \sqcup B = \{0, 1\}^n$, where A represents the points where μ “looks uniform”, $A := \{x : \mu(x) \in [1/(TN), T/N]\}$ for an appropriate $T > 1$. First we assume $\mathbf{Inf}_\mu[q] \leq \kappa n$ (otherwise we are done immediately), and then we then argue two facts:

- *A-part covering numbers.* The covering number of the A -part of q , *i.e.*, the set of $q_A = q \cdot \mathbf{1}_A$ for μ -compatible q , is suitably controlled by $\mathbf{Inf}_\mu[q]$. After some effort, this allows the ε -net argument to go through for q_A , and we have: w.h.p over $f \in F_\mu$, $\mathbf{Avg}_{\widehat{f}}[f q_A] \leq (1/2 + \text{small}) \|q_A\|_{L^2(\mu)}$ for all q_A corresponding to low-influence q .
- *B-part operator bounds.* Second, for the “bad” part of q , $q_B = q \cdot \mathbf{1}_B$ with $B = A^c$, we can afford to make a coarse bound because $|B|/N$ concentrates below a small constant of our choosing (depending on T). In particular, defining $\widehat{M}_{\widehat{f}}$ by the relation $\mathbf{Avg}_{\widehat{f}}[\widehat{g}] = \langle g, \widehat{M}_{\widehat{f}} g \rangle$, we show the operator norm of $\Pi_B \widehat{M}_{\widehat{f}} \Pi_B$ is small (close to $1/2$) with high probability. Equivalently, $\langle g_B, \widehat{M}_{\widehat{f}} g_B \rangle$ is small with high probability. The cross terms in $\mathbf{Avg}_{\widehat{f}}[(g_A + g_B)] = \langle g_A + g_B, \widehat{M}_{\widehat{f}} (g_A + g_B) \rangle$ are dispatched similarly, and are close to $1/4$ with high probability.

Together, these bounds yield the following schematic argument in the low $\mathbf{Inf}_\mu[q]$ case:

$$\begin{aligned}
\mathbf{Avg}_{\widehat{f}}[\widehat{g}] &= \langle g, \widehat{M}_{\widehat{f}} g \rangle \\
&= \langle g_A + g_B, \widehat{M}_{\widehat{f}} (g_A + g_B) \rangle \\
&\leq \mathbf{Avg}_{\widehat{f}}[f q_A] + 2 \|\Pi_A \widehat{M}_{\widehat{f}} \Pi_B\|_{\text{op}} \cdot \|g_A\|_2 \|g_B\|_2 + \|\Pi_B \widehat{M}_{\widehat{f}} \Pi_B\|_{\text{op}} \|g_B\|_2^2 \\
&\leq (1/2 + \text{small}) \|q_A\|_{L^2(\mu)}^2 + (1/2 + \text{small}) \|g_A\|_2 \|g_B\|_2 + (1/2 + \text{small}) \|g_B\|_2^2 \\
&= (1/2 + \text{small}) \|g_A\|_2^2 + (1/2 + \text{small}) \|g_A\|_2 \|g_B\|_2 + (1/2 + \text{small}) \|g_B\|_2^2 \\
&= (\|g_A\|_2, \|g_B\|_2) \begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} \|g_A\|_2 \\ \|g_B\|_2 \end{pmatrix} + \text{small} \\
&\leq 3/4 + \text{small}.
\end{aligned}$$

Here, the fourth line follows from the spectral bounds on $\Pi_A M_f \Pi_B$ and $\Pi_B M_f \Pi_B$ (plus the A -part analysis); the fifth line follows from the fact that $\|q_A\|_{L^2(\mu)} = \|g_A\|$; and the last line follows from the observations

$$\left\| \begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{pmatrix} \right\|_{\text{op}} = \frac{3}{4} \quad \text{and} \quad \left\| (\|g_A\|_2, \|g_B\|_2) \right\|_2 = \|g\|_2 = 1.$$

Combining this bound with the high-influence side of the win-win argument, we conclude $\mathbf{Avg}_f[g] + \mathbf{Avg}_{\widehat{f}}[\widehat{g}] \leq \max\{2 - \kappa, 1.75 + \text{small}\} \leq 2 - \Omega(1)$, as desired.

As anticipated, this uncertainty principle has a satisfying equivalent reformulation in terms of influences, as in [Theorem 2](#).

We conclude the technical overview with some technical details about the A - and B -part analyses.

A-part analysis: from influence on A to influence everywhere via linear extensions.

There is a significant—but subtle—issue with completing the covering number argument for low-influence q_A ’s. Recall that in the win-win argument we are only allowed to assume that $\mathbf{Inf}_\mu$ is small for q itself, but here we would like to control the covering number of the q_A ’s, and it is *not* always the case that $\mathbf{Inf}_\mu(q_A) \leq \mathbf{Inf}_\mu(q)$. This lack of monotonicity may be blamed on the edges crossing the A - B boundary, which can contribute very different values to $\mathbf{Inf}_\mu$ for q vs. q_A . The most that can be said immediately is that $\mathbf{Inf}_\mu^{\text{int}(A)}(q_A) \leq \mathbf{Inf}_\mu(q)$, where the *interior influence*

$$\mathbf{Inf}_\mu^{\text{int}(A)}(q_A) := \sum_{\substack{i, x \in A, \\ x+i \in A}} \frac{\mu(x)\mu(x+i)}{2\mu(x) + \mu(x+i)} \left(\frac{q(x) - q(x+i)}{2} \right)^2$$

is the subsum of $\mathbf{Inf}_\mu[q]$ using only edges fully inside A . And unfortunately, a bound on $\mathbf{Inf}_\mu^{\text{int}(A)}(q_A)$ alone does not always constrain the covering number of the q_A ’s—some counterexamples are explained in [Section 6](#).

The crucial insight is this: as long as μ is derived from a Haar-random f , the geometry necessary for these counterexamples to arise is not present except with very low probability. More formally, to finish the proof, we show that with high probability over the choice of μ , a bound on $\mathbf{Inf}_\mu^{\text{int}(A)}[q_A]$ actually *does* upper-bound the influence of a certain extension $\text{Ext}_A q_A$ of q_A to the whole cube, and this suffices to control the metric entropy of the q_A ’s. We make this comparison via a careful combinatorial argument, drawing some inspiration from the canonical paths analysis of [\[HPS25\]](#) and their “local escape property.”

B-part analysis: decoupled Fourier coefficients and random pavings. We explain the argument for bounding $\|\Pi_B \widehat{M}_{\widehat{f}} \Pi_B\|_{\text{op}}$; the cases of $\Pi_A \widehat{M}_{\widehat{f}} \Pi_B$ and $\Pi_B \widehat{M}_{\widehat{f}} \Pi_A$ are similar. First observe that, as μ varies, B acts as a random subset of $\{\pm 1\}^n$ with permutation-invariant law—and actually membership in B is very nearly i.i.d. Bernoulli-distributed. As such, Π_B can be considered a random coordinate projection, and when applied to a fixed matrix as in $\Pi_B Q \Pi_B$, constitutes a *random paving* in the sense of a work of Tropp [\[Tro08\]](#) concerning a partial resolution to the Kadison–Singer conjecture. Provided certain technical conditions hold, [\[Tro08\]](#) roughly states that sufficiently sparse random coordinate projections have small operator norm.

Of course, our situation is not $\Pi_B Q \Pi_B$ for fixed Q ; $\widehat{M}_{\widehat{f}}$ is certainly not independent of B . However, the entries of $\widehat{M}_{\widehat{f}}$ are defined by the Fourier coefficients of $\widehat{f}$, and it turns out that for Haar-random f , $\mathcal{O}(n)$ -sized marginals of $(\widehat{f}(s))_s$ can actually be considered as independent of μ . Thus for tracial moments of $\Pi_B \widehat{M}_{\widehat{f}} \Pi_B$ up to order $\mathcal{O}(n)$, we can “freeze” an approximation to $\widehat{M}_{\widehat{f}}$ and apply the random paving result of Tropp. Unfreezing $\widehat{M}_{\widehat{f}}$ gives the result. This argument is quite involved, and requires several Lindeberg-type replacement and Gaussian interpolation steps. Moreover, throughout, we must work with the τ -smoothed quantities introduced earlier.

2.6 Outlook

Our work leaves several questions open.

1. *Hay from the haystack.* Our test can handle all but a $2^{-\Omega(n)}$ Haar-fraction of target states. What familiar, structured families are in the “good” set? For example, the concurrent work

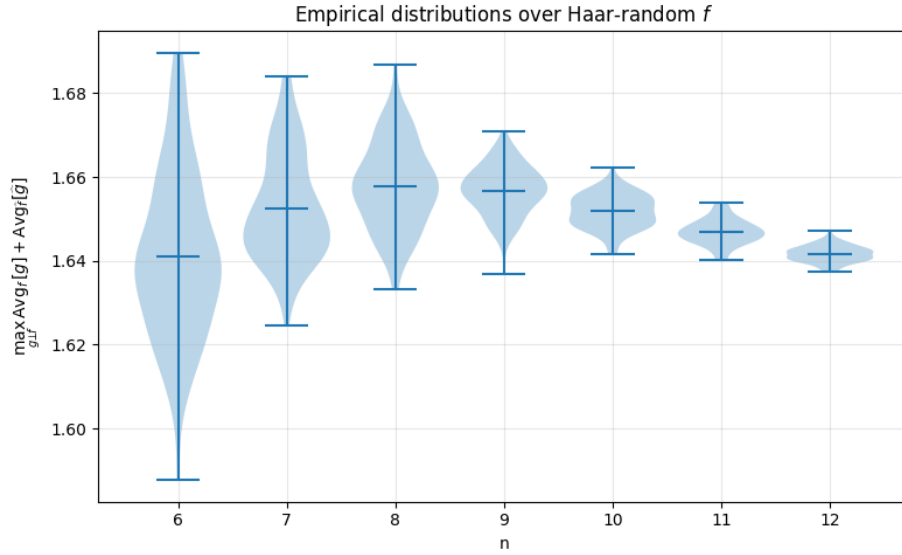


Figure 2: Numerical values for the uncertainty principle constant $2 - c$. Each distribution represents the empirical distribution of the second eigenvalue of $A_f + H^{\otimes n} A_{\hat{f}} H^{\otimes n}$ for 100 Haar-random samples.

by Du et al. [DLH⁺26], which also certifies almost all target states with constant robustness, provably applies to random graph states, a subclass of stabilizer states, and, numerically, seems to apply to random brickwork states (which we expect to be the case for our test too).

2. *Improving the robustness constant.* We proved that our test is C -robust, *i.e.* it can distinguish states that are ε -close from states that are $C\varepsilon$ -far, for some *very large* constant C closely related to the second eigenvalue of the $M_f + H^{\otimes n} M_{\hat{f}} H^{\otimes n}$ operator. However, experiments suggest that C should be at most 4—see Figure 2. The constant is important in practice: establishing that $C = 4$ means that our test can distinguish, *e.g.*, states that have 99.9 fidelity with the target from states that only have 99.6 fidelity). For comparison, in our prior work [CLSW26], although our test relied on one $\mathcal{O}(\log n)$ -qubit measurement, we were able to formally show that $C = 2$.
3. *A “fully tolerant” test.* One might also wish for an algorithm that is $(\varepsilon_1, \varepsilon_2)$ -tolerant for independently-settable $\varepsilon_1, \varepsilon_2 > 0$, meaning that it can distinguish between states that are ε_1 -close and states that are ε_2 -far. In order to achieve this, it is necessary to depart from the framework of a one-shot (independent of the tolerance parameters) test that is subsequently amplified sequentially.
4. *Robustness for all states.* As mentioned, our test can only handle all but a $2^{-\Omega(n)}$ Haar-fraction of target states. This limitation is inherent for non-adaptive tests: Gupta, He, and O’Donnell [GHO25] showed that some *adaptivity* is required for any test (robust or not) that handles *all* target states. Finding such a test, or showing that it does not exist, is one of the main open questions in this area.

3 Preliminaries and conventions

3.1 Notation

We briefly fix some notation used throughout. Further, special symbols defined throughout the proof are recorded in [Appendix A](#).

- We denote $N = 2^n$ always.
- We use the unitary normalization of the Fourier transform over $\mathbb{Z}_2^n$:

$$\widehat{f}(s) = \frac{1}{\sqrt{N}} \sum_x f(x) (-1)^{\sum_j s_j x_j}$$

- We write $s + t$ for $s, t \in \{0, 1\}^n$ to mean the bitwise XOR. If $i \in [n]$, we interpret $s + i$ as s with the i th index negated. The latter will also be written as $s + e_i$ when extra clarity is beneficial.
- The undecorated L^p norm $\|\cdot\|_p$ is always with respect to the *counting* measure.
- We denote $\langle f, g \rangle = \sum_x \overline{f(x)} g(x)$. We use $\langle f, Mg \rangle = \langle f | M | g \rangle$ interchangeably. For a probability measure μ on $\{0, 1\}^n$, we use $\langle f, g \rangle_\mu = \sum_x \mu(x) \overline{f(x)} g(x)$.
- The shorthand “unif” denotes the uniform probability distribution on $\{0, 1\}^n$.
- A *Steinhaus* random variable is a uniform (Haar) draw from the set $\{z \in \mathbb{C} : |z| = 1\}$.
- Exponents attach first. For example, for some random matrix A ,

$$\mathbf{E} \operatorname{tr} A^p := \mathbf{E}[\operatorname{tr} [A^p]]$$

- For a probability measure μ on the hypercube, define $\langle f, g \rangle_\mu = \sum_x \mu(x) \overline{f(x)} g(x)$.

3.2 Basic facts about the Porter–Thomas distribution

For any probability measure μ on $\{0, 1\}^n$ let the (*Fourier*) *bias* of μ be denoted by

$$\operatorname{Bias}(\mu) = \max_{s \neq 0^n} \sqrt{N} |\widehat{\mu}(s)| = \max_{s \neq 0^n} \left| \sum_x \mu(x) (-1)^{s \cdot x} \right|.$$

Lemma 1 (Bias of Porter-Thomas measures). *Let μ be Dirichlet($\vec{1}$) on $N = 2^n$ coordinates. Then with probability at least $1 - 2^{-\Omega(n)}$,*

$$\operatorname{Bias}(\mu) \leq \mathcal{O}\left(\sqrt{\frac{n}{N}}\right).$$

Proof. Fix a nonzero $s \in \{0, 1\}^n$. Write

$$E_s := \{x : \chi_s(x) = 1\}, \quad O_s := \{x : \chi_s(x) = -1\}.$$

Since $s \neq 0$, the character χ_s is balanced, and hence

$$|E_s| = |O_s| = N/2.$$

Therefore, using unitary Fourier normalization,

$$\sqrt{N} \widehat{\mu}(s) = \sum_x \mu(x) \chi_s(x) = \mu(E_s) - \mu(O_s) = 2\mu(E_s) - 1.$$

By the aggregation property of the Dirichlet distribution,

$$\mu(E_s) \sim \text{Beta}(N/2, N/2).$$

It is standard that $\text{Beta}(N/2, N/2)$ is subgaussian with scale $N^{-1/2}$ around $1/2$. Thus there is a universal constant $c > 0$ such that for all $t \geq 0$,

$$\Pr\left[\sqrt{N} |\hat{\mu}(s)| \geq t\right] = \Pr[|\mu(E_s) - 1/2| \geq t/2] \leq 2 \exp(-cNt^2).$$

Taking a union bound over all $N - 1$ nonzero s , we get

$$\Pr\left[\max_{s \neq 0} \sqrt{N} |\hat{\mu}(s)| \geq t\right] \leq 2N \exp(-cNt^2).$$

Now choose

$$t = C \sqrt{\frac{n}{N}}$$

with C sufficiently large. Since $N = 2^n$,

$$2N \exp(-cNt^2) = 2 \exp(n \log 2 - cC^2 n) \leq 2^{-\Omega(n)}$$

for C large enough. This proves the claim. $\square$

By similar, routine arguments, the same bound holds for the $3/2$ -norm:

Fact 1. *With probability at least $1 - \exp(-\Omega(n))$,*

$$\sum_x \mu(x)^{3/2} \leq \mathcal{O}\left(\sqrt{\frac{n}{N}}\right).$$

4 Reducing to an uncertainty principle

We begin the analysis of our one-shot test, [Protocol 1](#). From here on, the paper will focus on proving that [Protocol 1](#) satisfies [Theorem 1](#), restated here for convenience.

Theorem 1 (Constant robustness with single-qubit measurements). *There exists an efficient protocol using single-qubit measurements that, for all but a $2^{-\Omega(n)}$ fraction of pure n -qubit target states $|\psi\rangle$, satisfies the following. Given oracle access to a classical description of the target state $|\psi\rangle$ (via the model in [Definition 1](#)), and a single copy of an n -qubit mixed state ρ ,*

- (Completeness) *Outputs accept with probability at least $\langle \psi | \rho | \psi \rangle$.*
- (Soundness) *Outputs reject with probability at least $\frac{1 - \langle \psi | \rho | \psi \rangle}{C} - o(1)$, for some constant $C > 1$ (i.e. approximately the “infidelity” up to a constant factor).*

Moreover, the single-qubit measurements are all non-adaptive Pauli measurements, except for one final (non-Pauli) adaptive measurement.

The first observation is that, for a target state $|\psi\rangle$ and a lab state ρ , we can rewrite the acceptance probability in [Protocol 1](#) as $\text{Tr}[A\rho]$ where A is the following PSD operator:

$$A = \frac{1}{2n} \sum_i \sum_{z \in \{0,1\}^{n-1}} |z\rangle\langle z|_{[n]\setminus i} \otimes |\psi_z^{(i)}\rangle\langle\psi_z^{(i)}|_i + \frac{1}{2n} \sum_i \sum_{z \in \{0,1\}^{n-1}} (H^{\otimes n-1} |z\rangle\langle z| H^{\otimes n-1})_{[n]\setminus i} \otimes |\hat{\psi}_z^{(i)}\rangle\langle\hat{\psi}_z^{(i)}|_i,$$

where $|\psi_z^{(i)}\rangle$ is the post-measurement state of the i -th qubit upon measuring all qubits of $|\psi\rangle$, except the i -th, in the Z basis and obtaining outcome z ; and $|\widehat{\psi}_z^{(i)}\rangle$ is defined analogously, except the measurements are in the X basis.

We first argue *completeness*. Consider first a pure lab state $|\phi\rangle$, and write it in an eigenbasis of A . Notice then that $\text{Tr}[A|\phi\rangle\langle\phi|]$ is a convex combination of the contributions from each eigenvector of A (since the eigenvectors are orthogonal to each other and remain orthogonal under the action of A). Since $|\psi\rangle$ itself is an eigenvector of A with eigenvalue $+1$, we have $\text{Tr}[A|\phi\rangle\langle\phi|] \geq |\langle\psi|\phi\rangle|^2$. This extends to mixed lab states ρ , to give $\text{Tr}[A\rho] \geq \langle\psi|\rho|\psi\rangle$.

Thus, from here on, we will focus on *soundness*. For convenience, let us switch to a notation that will be more natural going forward: denote the target state by $|f\rangle = \sum_x f(x)|x\rangle$. Notice that it suffices to argue soundness for pure lab states $|g\rangle = \sum_x g(x)|x\rangle$, as soundness for mixed lab states ρ will follow by convexity. We can further reduce soundness to a statement about lab states $|g\rangle$ that are orthogonal to $|f\rangle$:

Proposition 4. *With all but $2^{-\Omega(n)}$ probability over a Haar-random target state $|f\rangle$, for all pure states $|g\rangle$ orthogonal to $|f\rangle$, we have*

$$\text{Tr}[A_f |g\rangle\langle g|] \leq 1 - c + o(1),$$

for some constant $c > 0$ where A_f is the acceptance operator corresponding to the target state $|f\rangle$.

Soundness of Theorem 1 from Proposition 4. We know $\langle f|A_f|f\rangle = 1$, and, by Proposition 4, with probability $1 - 2^{-\Omega(n)}$ over the target state $|f\rangle$, the following holds for all $|g\rangle$ orthogonal to $|f\rangle$: $\langle f|A_f|f\rangle \leq 1 - c + o(1)$, for some $c > 0$. Then, $A_f \preceq |f\rangle\langle f| + (1 - c + o(1))P$ for $P = I - |f\rangle\langle f|$. That is,

$$A_f \preceq (1 - c + o(1))I + (c - o(1))|f\rangle\langle f|.$$

Thus, any lab state ρ satisfies

$$\text{tr}[A_f \rho] \leq 1 - c + o(1) + (c - o(1))\langle f|\rho|f\rangle.$$

Rearranging gives

$$\Pr[\text{reject}] = 1 - \text{tr}[A_f \rho] \geq c(1 - \langle f|\rho|f\rangle) - o(1).$$

This yields exactly the desired soundness of Theorem 1 by setting $C = \frac{1}{c}$. $\square$

So, our goal for the rest of the paper will be to prove Proposition 4. As discussed in the technical overview, we will view Proposition 4 as an *uncertainty principle*. Let $|f\rangle = H^{\otimes n}|f\rangle$ and $|\widehat{g}\rangle = H^{\otimes n}|g\rangle$. Now, A_f is the sum of a “ Z -basis test” term and an “ X -basis test” term. It is straightforward to verify that we can rewrite $\text{Tr}[A_f |g\rangle\langle g|]$ as

$$\frac{1}{2}\mathbf{Avg}_f[g] + \frac{1}{2}\mathbf{Avg}_{\widehat{f}}[\widehat{g}],$$

where

$$\begin{aligned}\mathbf{Avg}_f[g] &:= \frac{1}{2n} \sum_i \sum_x \frac{\left| \overline{f(x)}g(x) + \overline{f(x+i)}g(x+i) \right|^2}{|f(x)|^2 + |f(x+i)|^2}, \\ \mathbf{Avg}_{\widehat{f}}[\widehat{g}] &:= \frac{1}{2n} \sum_i \sum_s \frac{\left| \widehat{f(s)}\widehat{g(s)} + \widehat{f(s+i)}\widehat{g(s+i)} \right|^2}{|\widehat{f(s)}|^2 + |\widehat{f(s+i)}|^2}.\end{aligned}$$

Here, for an index $i \in [n]$, we used the shorthand notation $x+i$ to denote the string where we flip the i -th bit of x . And the additional factor of $\frac{1}{2}$ in $\mathbf{Avg}_f[g]$ and $\mathbf{Avg}_{\widehat{f}}[\widehat{g}]$ comes from the fact that the sum over x double counts each inner product. Thus, we can equivalently rewrite our goal as the following theorem (restated from the technical overview):

Theorem 4 (The uncertainty principle). *With probability at least $1 - 2^{-\Omega(n)}$ over a Haar random $f : \{0, 1\}^n \rightarrow \mathbb{C}$, $\|f\|_2 = 1$, for all $g : \{0, 1\}^n \rightarrow \mathbb{C}$, $\|g\|_2 = 1$ with $\langle f, g \rangle = 0$,*

$$\mathbf{Avg}_f[g] + \mathbf{Avg}_{\widehat{f}}[\widehat{g}] \leq 2 - c,$$

where $c > 0$ is a universal constant independent of n .

As mentioned in the technical overview, the latter uncertainty principle has an elegant reformulation as an uncertainty principle for (generalized) total influences.

Corollary 2 (Uncertainty principle for generalized influences). *For a probability measure μ on $\{0, 1\}^n$ and a function $h : \{0, 1\}^n \rightarrow \mathbb{C}$, define the “Harmonic mean weighted influence”*

$$\mathbf{Inf}_\mu[h] = \sum_{i,x} \frac{2\mu(x)\mu(x+i)}{\mu(x) + \mu(x+i)} \left(\frac{h(x) - h(x+i)}{2} \right)^2$$

Then, with high probability over a Haar random $f : \{0, 1\}^n \rightarrow \mathbb{C}$, $\|f\|_2 = 1$, for all $g : \{0, 1\}^n \rightarrow \mathbb{C}$, $\|g\|_2 = 1$ with $\langle f, g \rangle = 0$,

$$\mathbf{Inf}_{\mu_f} \left[\frac{g}{f} \right] + \mathbf{Inf}_{\mu_{\widehat{f}}} \left[\frac{\widehat{g}}{\widehat{f}} \right] \geq cn,$$

where $c > 0$ is the same constant as in [Theorem 4](#).

Proof. This follows immediately from the identity

$$\mathbf{Avg}_f[g] = 1 - \frac{1}{n} \mathbf{Inf}_{\mu_f} \left[\frac{g}{f} \right],$$

which holds for all f and g such that $\|f\|_2 = 1$, $\|g\|_2 = 1$, and hence also for $\widehat{f}$ and $\widehat{g}$ (as long as f and $\widehat{f}$ are non-zero everywhere, which happens with probability 1). We leave the proof of this identity to [Appendix C](#). $\square$

5 Concentration of $\mathbf{Avg}_{\widehat{f}}[\widehat{f}q]$ for fixed q

The first key technical step in our argument (which is outlined at a high level in the technical overview) is to prove the following concentration bound for $\mathbf{Avg}_{\widehat{f}}[\widehat{f}q]$. This section is devoted to the proof of this bound.

Theorem 5 (Concentration of $\mathbf{Avg}_{\widehat{f}}[\widehat{f}q]$ for fixed q). *There exists a universal constant $c > 0$ such that for all $t > 0$, with probability $1 - \exp(-\Omega(n))$ over $\mu \sim \boldsymbol{\mu}$, for any μ -compatible q satisfying*

$$\max_x \mu(x) |q(x)|^2 \leq \|\mu\|_\infty^{1/3},$$

we have the bound

$$\Pr_{f \sim F_\mu} \left[\mathbf{Avg}_{\widehat{f}}[\widehat{f}q] \geq \frac{1}{2} + t + o(1) \right] \leq \exp\left(-ct^6 N^{1/3 - o(1)}\right).$$

Fixing f and g for now, let us establish some notation. Write

$$A^{(i)} = \frac{1}{2} \sum_s |\langle v_s, w_s \rangle|^2$$

for

$$v_s = \frac{u_s}{\|u_s\|_2} \quad \text{with} \quad u_s = (\widehat{f}(s), \widehat{f}(s+i)) \quad \text{and} \quad w_s = (\widehat{g}(s), \widehat{g}(s+i)).$$

Then $\mathbf{Avg}_{\widehat{f}}[\widehat{g}] = \frac{1}{n} \sum_i A^{(i)}$.

Each $A^{(i)}$ is a sum of terms each of which depend on disjoint pairs of Fourier coefficients of the underlying random vector $(f(x))_x$. If the underlying randomness were rotationally-invariant (namely, Gaussian), then unitarity of the Fourier transform would also imply $A^{(i)}$ is the sum of independent random variables, each functions of disjoint independent Fourier random variables. Unfortunately, the Steinhaus ensemble $(f(x))_x$ is not rotationally invariant. That would not be a problem if v_s 's were not normalized, for then $A^{(i)}$ would at least be a low-degree polynomial chaos. But normalized it is, and so typical approaches—even bounded differences or Herbst-type arguments—do not immediately apply.

To get around this, our strategy is to introduce for each $i \in [n]$ the following smoothed quantity, which has well-behaved derivatives. For $\delta > 0$, let

$$A_\delta^{(i)} := \frac{1}{2} \sum_s \frac{|\langle u_s, w_s \rangle|^2}{\|u_s\|_2^2 + \delta}.$$

Let $D_\delta^{(i)} = A^{(i)} - A_\delta^{(i)}$ be the difference. Now $D_\delta^{(i)}$ also has bad derivatives, so it would seem we have done nothing to help the situation. But in fact, $D_\delta^{(i)}$ is pointwise dominated by something nice:

$$\begin{aligned} D_\delta^{(i)} &= \frac{1}{2} \sum_s |\langle u_s, w_s \rangle|^2 \left(\frac{1}{\|u_s\|_2^2} - \frac{1}{\|u_s\|_2^2 + \delta} \right) = \frac{1}{2} \sum_s |\langle u_s, w_s \rangle|^2 \frac{\delta}{\|u_s\|_2^2 \cdot (\|u_s\|_2^2 + \delta)} \\ &\leq \frac{1}{2} \sum_s \frac{\delta \|w_s\|_2^2}{\|u_s\|_2^2 + \delta} \quad (\text{Cauchy-Schwarz}) \\ &=: B_\delta^{(i)}. \end{aligned}$$

The gradient of $B_\delta^{(i)}$ is sufficiently well-behaved for a Herbst-type argument succeed.

Theorem 5 is proved in two parts. First, we show the expectation is $1/2 + o(1)$ by passing to a Gaussian model (each $f(x)$ distributed as a complex Gaussian with variance $\mu(x)$). Then we show $\mathbf{Avg}_{\widehat{f}}[\widehat{f}q]$ concentrates about this expectation via a Herbst argument and log-Sobolev inequalities for Steinhaus ensembles. The δ -smoothing idea above will be crucial in both parts.

5.1 Technical preparations

We will need hypercontractivity for Steinhaus random variables:

Fact 2 (L^p - L^2 Steinhaus Hypercontractivity [Wei80]). *Let $\mathbb{T}^n = \{(z_1, \dots, z_n) : |z_i| = 1\}$. Let $f : \mathbb{T}^n \rightarrow \mathbb{C}$ be a polynomial in $z_i, \bar{z}_i$, $i = 1, \dots, n$ of total degree at most d . Then for all $p \geq 2$,*

$$(\mathbf{E} |f|^p)^{1/p} \leq (p-1)^{d/2} (\mathbf{E} |f|^2)^{1/2}.$$

And let us also state the straightforward Wick-type pairing rule about Steinhaus mixed moments.

Fact 3 (Steinhaus pairing rule). *Let $J \in \mathbb{N}$. For ω_j i.i.d. Steinhaus random variables, and a_j, b_j integers, for $j = 1, \dots, J$, one has*

$$\mathbf{E} \prod_j \omega_j^{a_j} \bar{\omega}_j^{b_j} = \begin{cases} 1 & \text{if } a_j = b_j, j = 1, \dots, J \\ 0 & \text{otherwise} \end{cases}$$

We will also use moments of w_s sums in several places.

Lemma 2. *Let $W = \sum_s \|w_s\|_2^4$. There exist universal constants $c, C > 0$ such that*

$$\mathbf{E} W \leq \frac{8}{N} \quad \text{and} \quad \Pr[|W - \mathbf{E} W| > t] \leq \exp(-c\sqrt{Nt}) \quad \text{for all } t \geq \frac{C}{N}.$$

In particular, we will use the instantiation

$$\Pr \left[W > \frac{K}{N^{1/3}} \right] \leq \exp(-c\sqrt{K}N^{1/3}). \quad (9)$$

Proof. We claim that for all s ,

$$\mathbf{E} \|w_s\|_2^4 \leq \frac{8}{N^2}. \quad (10)$$

Indeed,

$$\mathbf{E} \|w_s\|_2^4 = \mathbf{E} |\widehat{g}(s)|^4 + 2 \mathbf{E} |\widehat{g}(s)|^2 |\widehat{g}(s+i)|^2 + \mathbf{E} |\widehat{g}(s+i)|^4$$

and by [Fact 3](#), for any s ,

$$\begin{aligned} \mathbf{E} |\widehat{g}(s)|^4 &= \frac{1}{N^2} \mathbf{E} |\sum_x \sqrt{\mu(x)} \omega_x q(x) (-1)^{\langle x, s \rangle}|^4 \\ &= \frac{1}{N^2} \left(2 \sum_{x,y} \mu(x) |q(x)|^2 \mu(y) |q(y)|^2 - \sum_x \mu(x)^2 |q(x)|^4 \right) \\ &= \frac{1}{N^2} \left(2 - \sum_x \mu(x)^2 |q(x)|^4 \right) \leq \frac{2}{N^2}. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{E} |\widehat{g}(s)|^2 |\widehat{g}(s+i)|^2 &= \frac{1}{N^2} \left(1 + \left| \sum_x \mu(x) |q(x)|^2 (-1)^{x_i} \right|^2 - \sum_x \mu(x)^2 |q(x)|^4 \right) \\ &\leq \frac{1}{N^2} \left(1 + \left| \sum_x \mu(x) |q(x)|^2 \right|^2 \right) \quad (\|g\|_2^2 = 1) \\ &= \frac{2}{N^2}. \end{aligned}$$

This proves (10); summing over s then gives $\mathbf{E} W \leq 8/N$.

Towards concentration, applying Minkowski (L^2 triangle inequality) we get

$$\text{Var}[W] \leq \mathbf{E} W^2 = \mathbf{E} \left(\sum_s \|w_s\|_2^4 \right)^2 \leq \left(\sum_s (\mathbf{E} \|w_s\|_2^8)^{1/2} \right)^2.$$

Then by hypercontractivity for Steinhaus ([Fact 2](#)) and Eq. (10),

$$\mathbf{E} \|w_s\|_2^8 \lesssim (\mathbf{E} \|w_s\|_2^4)^2 \lesssim \frac{1}{N^4}.$$

So

$$\text{Var}[W] \lesssim \frac{1}{N^2}.$$

Markov's inequality finishes the argument. With p to be chosen later,

$$\begin{aligned} \Pr[|W - \mathbf{E} W| > t] &= \Pr[|W - \mathbf{E} W|^p > t^p] \\ &\leq \frac{\mathbf{E}[|W - \mathbf{E} W|^p]}{t^p} \\ &\leq t^{-p} (p-1)^{2p} \text{Var}(W)^{p/2} \quad (\text{Fact 2}) \\ &\leq \left(\frac{p^2 C}{tN} \right)^p \end{aligned}$$

and choosing $p = \sqrt{tN/(eC)}$,

$$\Pr[|W - \mathbf{E} W| > t] \leq \exp(-c\sqrt{tN}). \quad \square$$

5.2 Bounding the expectation

In this section we prove:

Lemma 3. *For all $t > 0$, for all n large enough, with exponential probability over $\mu \sim \boldsymbol{\mu}$, for all fixed μ -compatible q ,*

$$\mathbf{E}_{f \sim F_\mu} \mathbf{Avg}_{\widehat{f}}[\widehat{fq}] = \mathbf{E}_{f \sim F_\mu} \left[\frac{1}{n} \sum_i A^{(i)} \right] \leq \frac{1}{2} + t.$$

To evaluate the expectation, the general idea is to use that in the $n \rightarrow \infty$ limit, small marginals of the joint distribution $(\widehat{f}(s))_{s \in \{0,1\}^n}$ are jointly Gaussian—and each $A^{(i)}$ can be exactly integrated in the Gaussian model.

More formally, let $\mu \sim \boldsymbol{\mu}$ be fixed. In the Gaussian model we replace f with independent scaled Steinhaus entries with f sampled with independent entries $f(x) \sim \sqrt{\mu(x)} \cdot \mathcal{CN}(0, 1)$. Denote this distribution of f 's by $\text{Gauss}(\mu)$.

Unfortunately, while the $A^{(i)}$ are sums of terms involving only a small marginal of the joint distribution $(\widehat{f}(s))_s$, these terms are not themselves Lipschitz in the underlying randomness, so standard Berry–Esseen-type comparison theorems do not apply. To get the desired convergence to the Gaussian model, we must therefore work with the smoothed quantities $A_\delta^{(i)}$ and the error bounds $B_\delta^{(i)}$ introduced earlier.

In particular, because

$$\mathbf{E} A^{(i)} \leq \mathbf{E} A_\delta^{(i)} + \mathbf{E} B_\delta^{(i)},$$

Lemma 3 would be true if we could show the RHS terms are at most $1/2 + o(1)$ and $o(1)$ respectively. Schematically, we get the bound via

$$\begin{aligned} \mathbf{E} A^{(i)} &\leq \mathbf{E} A_\delta^{(i)} + \mathbf{E} B_\delta^{(i)} \\ &= \mathbf{E}_{\text{Gauss}(\mu)} A_\delta^{(i)} + \left(\mathbf{E} A_\delta^{(i)} - \mathbf{E}_{\text{Gauss}(\mu)} A_\delta^{(i)} \right) + \mathbf{E} B_\delta^{(i)} \\ &\leq \mathbf{E}_{\text{Gauss}(\mu)} A^{(i)} + \left(\mathbf{E} A_\delta^{(i)} - \mathbf{E}_{\text{Gauss}(\mu)} A_\delta^{(i)} \right) + \mathbf{E} B_\delta^{(i)} \end{aligned}$$

where the last line holds because $A_\delta^{(i)} \leq A^{(i)}$ pointwise. Each of the terms in the last line are bounded in a separate subsection: the first term is proved by direct integration, where we make heavy use of Gaussianity; the second term is by a Lindeberg replacement argument; and the third term is handled via negative moment bounds for Steinhaus sums.

5.2.1 Bounding $\mathbf{E} A^{(i)}$ in the Gaussian model

The main claim of this section is as follows:

Lemma 4. *With probability $1 - 2^{-\Omega(n)}$ over $\mu \sim \boldsymbol{\mu}$,*

$$\mathbf{E}_{\text{Gauss}(\mu)} \left[\frac{1}{n} \sum_i A^{(i)} \right] \leq \frac{1}{2} + o(1)$$

Let us record some facts about u_s and w_s in the Gaussian model. Begin by noting that for all s ,

$$u_s = \mathcal{CN}(0, \Sigma_i) \quad \text{with} \quad \Sigma_i = \frac{1}{N} \begin{pmatrix} 1 & \sqrt{N} \widehat{\mu}(i) \\ \sqrt{N} \widehat{\mu}(i) & 1 \end{pmatrix}.$$

Justification for the off-diagonal values of Σ_i is as follows:

$$\begin{aligned}\mathbf{E}[\widehat{f(s)}\widehat{f(s+i)}] &= \frac{1}{N} \mathbf{E}\left[\sum_{x,y} \overline{f(x)} f(y) \chi_s(x) \chi_{s+i}(y)\right] \\ &= \frac{1}{N} \mathbf{E}[\sum_x |f(x)|^2 \chi_i(x)] \\ &= \frac{1}{N} \sum_x \mu(x) \chi_i(x) \\ &= \frac{\widehat{\mu}(i)}{\sqrt{N}}.\end{aligned}$$

And similarly,

$$w_s = \mathcal{CN}(0, \widetilde{\Sigma}_i) \quad \text{with} \quad \widetilde{\Sigma}_i = \mathbf{E}[|w_s\rangle\langle w_s|] = \frac{1}{N} \begin{pmatrix} 1 & \sqrt{N} \widehat{\mu|q|^2}(i) \\ \sqrt{N} \widehat{\mu|q|^2}(i) & 1 \end{pmatrix}$$

Moreover, in this model we may relate w_s to u_s via Gaussian regression. That is, we may decompose

$$w_s = Au_s + \gamma, \quad A = R\Sigma^{-1}, \quad \gamma \perp u, \quad R = \mathbf{E}[w_s u_s^\dagger] = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & \widehat{q\mu}(i) \\ \widehat{q\mu}(i) & 0 \end{pmatrix}. \quad (11)$$

Note that the diagonal is zero because it is $\widehat{q\mu}(0) = \mathbf{E}_\mu q = 0$.

Before continuing, let us record a similar result for the expected outer product of v_s , *i.e.*, $\mathbf{E}[v_s v_s^\dagger]$:

Proposition 5. *Suppose $\sqrt{N} |\widehat{\mu}(i)| \leq 1/2$. Then*

$$\mathbf{E}[|v_s\rangle\langle v_s|] = \frac{1}{2} \begin{pmatrix} 1 & \Phi(\sqrt{N} \widehat{\mu}(i)) \\ \Phi(\sqrt{N} \widehat{\mu}(i)) & 1 \end{pmatrix},$$

where

$$\Phi(x) := \int_0^1 \frac{2t-1+x}{1+x(2t-1)} dt = \frac{2x - (1-x^2) \log \frac{1+x}{1-x}}{2x^2}.$$

Consequently,

$$\mathbf{E}[|v_s\rangle\langle v_s|] = \frac{1}{2} \begin{pmatrix} 1 & \mathcal{O}(\sqrt{N} \widehat{\mu}(i)) \\ \mathcal{O}(\sqrt{N} \widehat{\mu}(i)) & 1 \end{pmatrix}.$$

The proof is by explicit integration.

Proof. Let $\rho_i = \sqrt{N} \widehat{\mu}(i)$. The Hadamard matrix $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ diagonalizes Σ_i ; *i.e.*,

$$H\Sigma_i H = \frac{1}{N} \begin{pmatrix} 1 + \rho_i & 0 \\ 0 & 1 - \rho_i \end{pmatrix}.$$

So if $\xi_1, \xi_2 \stackrel{\text{iid}}{\sim} \mathcal{CN}(0, 1)$ and

$$y := \begin{pmatrix} \sqrt{1+\rho_i} \xi_1 \\ \sqrt{1-\rho_i} \xi_2 \end{pmatrix},$$

then u_s has the same distribution as $\frac{1}{\sqrt{N}} Hy$. Since normalization cancels the factor $1/\sqrt{N}$,

$$|v_s\rangle\langle v_s| \stackrel{(\text{dist})}{=} H \frac{yy^*}{\|y\|_2^2} H \quad \text{and} \quad \mathbf{E}[|v_s\rangle\langle v_s|] = H \mathbf{E}\left[\frac{yy^*}{\|y\|_2^2}\right] H.$$

Now write

$$X := |\xi_1|^2, \quad Y := |\xi_2|^2.$$

Then X, Y are i.i.d. $\text{Exp}(1)$. By phase symmetry,

$$\mathbf{E}\left[\frac{\xi_1 \bar{\xi}_2}{(1 + \rho_i)X + (1 - \rho_i)Y}\right] = 0,$$

so

$$\mathbf{E}\left[\frac{yy^*}{\|y\|_2^2}\right] = \begin{pmatrix} \phi(\rho_i) & 0 \\ 0 & 1 - \phi(\rho_i) \end{pmatrix},$$

where

$$\phi(\rho) := \mathbf{E}\left[\frac{(1 + \rho)X}{(1 + \rho)X + (1 - \rho)Y}\right].$$

Conjugating by H gives

$$\mathbf{E}[|v_s\rangle\langle v_s|] = \frac{1}{2} \begin{pmatrix} 1 & 2\phi(\rho_i) - 1 \\ 2\phi(\rho_i) - 1 & 1 \end{pmatrix}.$$

Thus it remains to identify

$$\Phi(\rho) := 2\phi(\rho) - 1 = \mathbf{E}\left[\frac{(1 + \rho)X - (1 - \rho)Y}{(1 + \rho)X + (1 - \rho)Y}\right].$$

Use the change of variables

$$r = X + Y, \quad t = \frac{X}{X + Y}.$$

For i.i.d. exponentials X, Y , $r \sim \Gamma(2, 1)$, $t \sim \text{Unif}[0, 1]$, and r, t are independent. Since $X = rt$ and $Y = r(1 - t)$, the r cancels and we get

$$\Phi(\rho) = \int_0^1 \frac{(1 + \rho)t - (1 - \rho)(1 - t)}{(1 + \rho)t + (1 - \rho)(1 - t)} dt = \int_0^1 \frac{2t - 1 + \rho}{1 + \rho(2t - 1)} dt. \quad \square$$

Lemma 4 is proved in two parts. First an upper bound on each $\mathbf{E}_{\text{Gauss}(\mu)} A^{(i)}$ is obtained in terms of quantities like $\widehat{\mu(i)}$, and then the average of these quantities over $i \in [n]$ is shown to be small.

Proposition 6. *For all μ with $\sqrt{N}|\widehat{\mu}(i)| \leq 1/2$ and all μ -compatible q , in the Gaussian model just described,*

$$0 \leq \mathbf{E}_{f \sim \text{Gauss}(\mu)}[A^{(i)}] \leq \frac{1}{2} + \mathcal{O}\left(\sqrt{N}|\widehat{\mu}(i)| + \frac{N|\widehat{\mu}q(i)|^2}{(1 - \sqrt{N}|\widehat{\mu}(i)|)^2}\right).$$

Proof. According to the Gaussian regression identities (11), the s^{th} term of $A^{(i)}$ decomposes as

$$|\langle v_s, w_s \rangle|^2 = |\langle v_s | A | u_s \rangle|^2 + |\langle v_s | \gamma \rangle|^2 + 2\Re\langle v_s | A | u_s \rangle \langle v_s | \gamma \rangle$$

And by independence,

$$\mathbf{E}[|\langle v_s, w_s \rangle|^2] = \underbrace{\mathbf{E}[|\langle v_s | A | u_s \rangle|^2]}_{(*)} + \underbrace{\mathbf{E}[|\langle v_s | \gamma \rangle|^2]}_{(**)}.$$

*Bound the (**) term.* Again by independence we have

$$(**) = \text{tr}(\mathbf{E}[|\gamma\rangle\langle\gamma|] \mathbf{E}[|v_s\rangle\langle v_s|]).$$

Because the covariance matrix $\tilde{\Sigma}_i$ of w_s dominates that of γ , we have $\mathbf{E}[|\gamma\rangle\langle\gamma|] \preceq \tilde{\Sigma}_i$. Therefore, using [Proposition 5](#),

$$\begin{aligned}
(**) &\leq \text{tr} \left(\tilde{\Sigma}_i \mathbf{E}[|v_s\rangle\langle v_s|] \right) \\
&= \text{tr} \left[\frac{1}{N} \begin{pmatrix} 1 & \sqrt{N} \widehat{\mu|q|^2}(i) \\ \sqrt{N} \widehat{\mu|q|^2}(i) & 1 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 & \Phi(\sqrt{N} \widehat{\mu}(i)) \\ \Phi(\sqrt{N} \widehat{\mu}(i)) & 1 \end{pmatrix} \right] \\
&= \frac{1}{N} \left[1 + \sqrt{N} \widehat{\mu|q|^2}(i) \Phi(\sqrt{N} \widehat{\mu}(i)) \right] \\
&\leq \frac{1}{N} \left(1 + C\sqrt{N} |\widehat{\mu}(i)| \right).
\end{aligned}$$

Here we used

$$\left| \sqrt{N} \widehat{\mu|q|^2}(i) \right| = \left| \sum_x \mu(x) |q(x)|^2 (-1)^{x_i} \right| \leq \sum_x \mu(x) |q(x)|^2 = 1,$$

together with

$$|\Phi(\rho)| \leq C|\rho| \quad \text{for } |\rho| \leq \frac{1}{2}. \quad \diamond$$

Bounding the $()$ term.*

$$(*) = \mathbf{E} \left[\left| \frac{\langle u_s | A | u_s \rangle}{\|u_s\|_2} \right|^2 \right] \leq \|A\|_{\text{op}}^2 \mathbf{E} \|u_s\|^2 = \|A\|_{\text{op}}^2 \text{tr} \Sigma_i = \frac{2}{N} \|A\|_{\text{op}}^2$$

Now to bound $\|A\|_{\text{op}}$, use that $\|R\|_{\text{op}} = |\widehat{\mu q}(i)|/\sqrt{N}$ and $\|\Sigma^{-1}\|_{\text{op}} = N/(1 - \sqrt{N}|\widehat{\mu}(i)|)$. Thus

$$(*) \leq \frac{2|\widehat{\mu q}(i)|^2}{(1 - \sqrt{N}|\widehat{\mu}(i)|)^2} \quad \diamond$$

Note that in both parts dependence on s has disappeared. Substituting back into $A^{(i)}$ we find

$$\begin{aligned}
\mathbf{E}[A^{(i)}] &\leq \frac{N}{2} \left[\frac{1}{N} \left(1 + C\sqrt{N} |\widehat{\mu}(i)| \right) + \frac{2|\widehat{\mu q}(i)|^2}{(1 - |\widehat{\mu}(i)|\sqrt{N})^2} \right] \\
&= \frac{1}{2} + \mathcal{O} \left(\sqrt{N} |\widehat{\mu}(i)| + \frac{N|\widehat{\mu q}(i)|^2}{(1 - \sqrt{N}|\widehat{\mu}(i)|)^2} \right). \quad \square
\end{aligned}$$

It remains to show that both $\widehat{\mu}(i)$ and $\widehat{\mu q}(i)^2$ are small with high probability:

Proposition 7. *With probability $1 - 2^{-\Omega(n)}$ over $\mu \sim \boldsymbol{\mu}$, for all μ -compatible q ,*

$$\frac{1}{n} \sum_i \left(\sqrt{N} |\widehat{\mu}(i)| + \frac{N|\widehat{\mu q}(i)|^2}{(1 - \sqrt{N}|\widehat{\mu}(i)|)^2} \right) = \mathcal{O}(1/n)$$

Proof. By [Lemma 1](#), with probability $1 - 2^{-\Omega(n)}$, $\sqrt{N} \widehat{\mu}(i) \leq \mathcal{O}(\sqrt{n/N})$. Thus it remains to understand the numerator of the second term in each summand. Summing over i , we find

$$\sum_i N |\widehat{\mu q}(i)|^2 = \sum_i |\langle q, \chi_i \rangle_\mu|^2 \leq \|G\|_{\text{op}} \|q\|_{L^2(\mu)}^2 = \|G\|_{\text{op}}$$

where G is Gram matrix of level-1 Fourier characters wrt μ inner product:

$$G_{ij} = \langle \chi_i, \chi_j \rangle_\mu.$$

Now write $G = \mathbf{1} + M$ and note that

$$\|M\|_{\text{op}} \leq \max_i \sum_{j \neq i} |M_{ij}| \leq n \max_{|s|=2} \sqrt{N} |\hat{\mu}(s)|,$$

where in the last inequality we have used that for $i \neq j$,

$$|\langle \chi_i, \chi_j \rangle| = |\sum_x \mu(x) (-1)^{x_i + x_j}| = \sqrt{N} |\hat{\mu}(\{i, j\})|.$$

By [Lemma 1](#), with high probability, $\max_{|s|=2} \sqrt{N} |\hat{\mu}(s)| = \mathcal{O}(\sqrt{n/N})$. Thus we conclude

$$\frac{1}{n} \sum_i N |\widehat{\mu q}(i)|^2 \leq \frac{1}{n} \|G\|_{\text{op}} \leq \frac{1}{n} (1 + n \mathcal{O}(\sqrt{n/N})) = \mathcal{O}(1/n). \quad \square$$

We are ready to finish this subsection.

Proof of [Lemma 4](#). Combining the propositions, we have

$$\frac{1}{n} \sum_i \mathbf{E}_{\text{Gauss}(\mu)} [A^{(i)}] \leq \frac{1}{2} + \mathcal{O} \left(\frac{1}{n} \sum_i \left(\sqrt{N} |\hat{\mu}(i)| + \frac{N |\widehat{\mu q}(i)|^2}{(1 - \sqrt{N} |\hat{\mu}(i)|)^2} \right) \right) = \frac{1}{2} + \mathcal{O}(1/n).$$

□

5.2.2 Comparing $\mathbf{E} A_\delta^{(i)}$ and $\mathbf{E}_{\text{Gauss}} A_\delta^{(i)}$

In this section it will be convenient to use the notation of Fréchet derivatives to differentiate random variables as functions with respect to a coordinate of underlying randomness in $\mathbb{C} \cong \mathbb{R}^2$. For any $h : \mathbb{C} \cong \mathbb{R}^2 \rightarrow \mathbb{R}$, the notation $D^J h(z)[v_1, \dots, v_J]$ denotes the J^{th} Fréchet derivative at z evaluated along directions $v_1, \dots, v_J \in \mathbb{C} \cong \mathbb{R}^2$ and extends to mixed derivatives in the natural way. In the context of Fréchet derivatives we shall also use the undecorated norm $\|\cdot\|$ to denote the operator norm of the relevant space. Concretely,

$$\|D^J h(z)\| := \sup_{\substack{v_j \in \mathbb{R}^2, \|v_j\|_2=1 \\ j=1,2,\dots,J}} |D^J h(z)[v_1, \dots, v_J]|.$$

For this section we will also use Φ_δ to describe the terms in our smoothed average:

$$\Phi_\delta(u, w) := \frac{|\langle u, w \rangle|^2}{\|u\|_2^2 + \delta}$$

We will need some bounds on the mixed derivatives of $\Phi_\delta(u, w)$; for this it is most convenient to argue in two steps.

Lemma 5 (Master derivative lemma). *Define*

$$P_\delta(u) := \frac{uu^*}{\|u\|_2^2 + \delta}, \quad u \in \mathbb{C}^2.$$

View $P_\delta : \mathbb{C}^2 \cong \mathcal{R}^4 \rightarrow M_2(\mathbb{C})$ as a smooth real Fréchet map. Then for each $k = 0, 1, 2, 3$,

$$\|D^k P_\delta(u)\| \lesssim_k \delta^{-k/2} \quad \text{for all } u \in \mathbb{C}^2.$$

Proof. Write

$$s(u) := \|u\|_2^2 + \delta, \quad T(u) := uu^*, \quad P_\delta(u) = \frac{T(u)}{s(u)}.$$

We regard all derivatives as real Fréchet derivatives on $\mathbb{C}^2 \cong \mathbb{R}^4$.

For $h, h_1, h_2, h_3 \in \mathbb{C}^2$, define

$$\beta_u(h) := 2\Re\langle u, h \rangle, \quad \gamma(h_1, h_2) := 2\Re\langle h_1, h_2 \rangle.$$

Then

$$DT(u)[h] = uh^* + hu^*, \quad D^2T(u)[h_1, h_2] = h_1h_2^* + h_2h_1^*, \quad D^3T(u) \equiv 0,$$

and

$$Ds(u)[h] = \beta_u(h), \quad D^2s(u)[h_1, h_2] = \gamma(h_1, h_2), \quad D^3s(u) \equiv 0.$$

Now let $r(u) := s(u)^{-1}$. Then

$$Dr(u)[h] = -\frac{\beta_u(h)}{s(u)^2},$$

$$D^2r(u)[h_1, h_2] = \frac{2\beta_u(h_1)\beta_u(h_2)}{s(u)^3} - \frac{\gamma(h_1, h_2)}{s(u)^2},$$

and

$$\begin{aligned} D^3r(u)[h_1, h_2, h_3] = & -\frac{6\beta_u(h_1)\beta_u(h_2)\beta_u(h_3)}{s(u)^4} \\ & + \frac{2(\gamma(h_1, h_2)\beta_u(h_3) + \gamma(h_1, h_3)\beta_u(h_2) + \gamma(h_2, h_3)\beta_u(h_1))}{s(u)^3}. \end{aligned}$$

Since $P_\delta = Tr$, the product rule gives

$$DP_\delta(u)[h] = \frac{uh^* + hu^*}{s(u)} - \frac{\beta_u(h)}{s(u)^2}uu^*.$$

Differentiating once more,

$$\begin{aligned} D^2P_\delta(u)[h_1, h_2] = & \frac{h_1h_2^* + h_2h_1^*}{s(u)} \\ & - \frac{\beta_u(h_2)(uh_1^* + h_1u^*) + \beta_u(h_1)(uh_2^* + h_2u^*)}{s(u)^2} \\ & - \frac{\gamma(h_1, h_2)}{s(u)^2}uu^* + \frac{2\beta_u(h_1)\beta_u(h_2)}{s(u)^3}uu^*. \end{aligned}$$

Differentiating a third time yields

$$\begin{aligned} D^3P_\delta(u)[h_1, h_2, h_3] = & -\frac{\beta_u(h_3)(h_1h_2^* + h_2h_1^*) + \beta_u(h_2)(h_1h_3^* + h_3h_1^*) + \beta_u(h_1)(h_2h_3^* + h_3h_2^*)}{s(u)^2} \\ & - \frac{\gamma(h_2, h_3)(uh_1^* + h_1u^*) + \gamma(h_1, h_3)(uh_2^* + h_2u^*) + \gamma(h_1, h_2)(uh_3^* + h_3u^*)}{s(u)^2} \\ & + \frac{2\beta_u(h_2)\beta_u(h_3)(uh_1^* + h_1u^*) + 2\beta_u(h_1)\beta_u(h_3)(uh_2^* + h_2u^*) + 2\beta_u(h_1)\beta_u(h_2)(uh_3^* + h_3u^*)}{s(u)^3} \\ & + \frac{2(\gamma(h_1, h_2)\beta_u(h_3) + \gamma(h_1, h_3)\beta_u(h_2) + \gamma(h_2, h_3)\beta_u(h_1))}{s(u)^3}uu^* \\ & - \frac{6\beta_u(h_1)\beta_u(h_2)\beta_u(h_3)}{s(u)^4}uu^*. \end{aligned}$$

We now estimate these in operator norm. Assume $\|h\|_2, \|h_j\|_2 \leq 1$. Then

$$\|uh^* + hu^*\| \leq 2\|u\|_2, \quad \|h_1h_2^* + h_2h_1^*\| \leq 2, \quad \|uu^*\| = \|u\|_2^2,$$

and

$$|\beta_u(h)| \leq 2\|u\|_2, \quad |\gamma(h_1, h_2)| \leq 2.$$

Therefore

$$\|DP_\delta(u)\| \lesssim \frac{\|u\|_2}{s(u)} + \frac{\|u\|_2^3}{s(u)^2},$$

$$\|D^2P_\delta(u)\| \lesssim \frac{1}{s(u)} + \frac{\|u\|_2^2}{s(u)^2} + \frac{\|u\|_2^4}{s(u)^3},$$

and

$$\|D^3P_\delta(u)\| \lesssim \frac{\|u\|_2}{s(u)^2} + \frac{\|u\|_2^3}{s(u)^3} + \frac{\|u\|_2^5}{s(u)^4}.$$

Using the elementary bound

$$\frac{\|u\|_2^m}{(\|u\|_2^2 + \delta)^\ell} \lesssim_{m,\ell} \delta^{m/2-\ell} \quad (m \leq 2\ell),$$

we obtain

$$\|DP_\delta(u)\| \lesssim \delta^{-1/2}, \quad \|D^2P_\delta(u)\| \lesssim \delta^{-1}, \quad \|D^3P_\delta(u)\| \lesssim \delta^{-3/2}.$$

Also,

$$\|P_\delta(u)\| = \frac{\|u\|_2^2}{\|u\|_2^2 + \delta} \leq 1.$$

This proves the lemma. □

Proposition 8. *Let*

$$\Phi_\delta(u, w) := \frac{|\langle u, w \rangle|^2}{\|u\|_2^2 + \delta}.$$

Then

$$\|D_u^3\Phi_\delta(u, w)\| \lesssim \delta^{-3/2}\|w\|_2^2, \quad \|D_u^2D_w\Phi_\delta(u, w)\| \lesssim \delta^{-1}\|w\|_2, \quad \|D_uD_w^2\Phi_\delta(u, w)\| \lesssim \delta^{-1/2},$$

and $D_w^3\Phi_\delta(u, w) \equiv 0$.

Proof. Write

$$\Phi_\delta(u, w) = w^*P_\delta(u)w.$$

Since Φ_δ is quadratic in w , we immediately have

$$D_w^3\Phi_\delta(u, w) \equiv 0.$$

Also,

$$D_w\Phi_\delta(u, w)[k] = 2\Re\langle P_\delta(u)w, k \rangle, \quad D_w^2\Phi_\delta(u, w)[k_1, k_2] = 2\Re\langle P_\delta(u)k_1, k_2 \rangle.$$

Hence

$$D_uD_w^2\Phi_\delta(u, w)[h, k_1, k_2] = 2\Re\langle DP_\delta(u)[h], k_1, k_2 \rangle,$$

so

$$\|D_uD_w^2\Phi_\delta(u, w)\| \lesssim \|DP_\delta(u)\| \lesssim \delta^{-1/2}$$

by [Lemma 5](#).

Similarly,

$$D_u^2 D_w \Phi_\delta(u, w)[h_1, h_2, k] = 2\Re\langle D^2 P_\delta(u)[h_1, h_2]w, k \rangle,$$

and therefore

$$\|D_u^2 D_w \Phi_\delta(u, w)\| \lesssim \|D^2 P_\delta(u)\| \|w\|_2 \lesssim \delta^{-1} \|w\|_2.$$

Finally,

$$D_u^3 \Phi_\delta(u, w)[h_1, h_2, h_3] = w^* D^3 P_\delta(u)[h_1, h_2, h_3]w,$$

whence

$$\|D_u^3 \Phi_\delta(u, w)\| \lesssim \|D^3 P_\delta(u)\| \|w\|_2^2 \lesssim \delta^{-3/2} \|w\|_2^2.$$

This proves the proposition. $\square$

With these technical facts established, we can move on to the main comparison of this subsection.

Proposition 9 (Smooth replacement). *For every fixed $i \in [n]$ and every $\delta = \eta/N$,*

$$\left| \mathbf{E}_{f \sim F_\mu} A_\delta^{(i)} - \mathbf{E}_{\text{Gauss}(\mu)} A_\delta^{(i)} \right| \lesssim_\eta \|\mu\|_\infty^{1/2}.$$

Proof. We make a Lindeberg replacement argument. Let $\xi_j \stackrel{\text{iid}}{\sim} \mathcal{CN}(0, 1)$, $j \in [N]$ and $\omega_j \stackrel{\text{iid}}{\sim} \text{Stein}$, $j \in [N]$. We will identify these indices j with points $x \in \{0, 1\}^n$; any bijection $\pi : [N] \rightarrow \{0, 1\}^n$ will do. Define the “hybrid” joint distributions

$$Z^{(m)} = (\xi_1, \dots, \xi_m, \omega_{m+1}, \dots, \omega_N).$$

Let us consider $A_\delta^{(i)}$ as an explicit function of the underlying randomness in any of these hybrids: write $A_\delta^{(i)}(Z^{(m)})$ to denote $A_\delta^{(i)}$ according to $f(\pi(k)) = \sqrt{\mu(\pi(k))}\xi_k$ for all $k \leq m$ and $f(\pi(k)) = \sqrt{\mu(\pi(k))}\omega_k$ for $k > m$. Then

$$\mathbf{E}_{f \sim F_\mu} A_\delta^{(i)} - \mathbf{E}_{\text{Gauss}(\mu)} A_\delta^{(i)} = \mathbf{E} A_\delta^{(i)}(Z^{(0)}) - \mathbf{E} A_\delta^{(i)}(Z^{(N)}) = \sum_{m=1}^N \mathbf{E} \left[A_\delta^{(i)}(Z^{(m-1)}) - A_\delta^{(i)}(Z^{(m)}) \right],$$

so it suffices to control the error of the increments.

To that end, let us fix m , freeze variables except those for the m^{th} coordinate and define

$$\begin{aligned} h : \mathbb{C} \cong \mathbb{R}^2 &\rightarrow \mathbb{R} \\ z &\mapsto A_\delta^{(i)}(\xi_1, \dots, \xi_{m-1}, z, \omega_{m+1}, \dots, \omega_N). \end{aligned}$$

Taking the second-order Taylor expansion of h at 0, we get

$$h(z) = h(0) + Dh(0)[z] + \frac{1}{2} D^2 h(0)[z, z] + R(z).$$

The remainder R satisfies

$$R(z) = \int_0^1 \frac{(1-t)^2}{2} D^3 h(tz)[z, z, z] dt, \quad \text{so} \quad |R(z)| \leq \frac{1}{2} |z|^3 \sup_{0 \leq t \leq 1} \|D^3 h(tz)\|.$$

Viewed as random vectors in $\mathbb{R}^2$, the ξ 's and ω 's have matching means and covariance matrices, so the constant, linear, and quadratic terms have the same expectation. Thus:

$$\begin{aligned} |\mathbf{E}[h(\omega_m) - h(\xi_m)]| &= |\mathbf{E} R(\omega_m) - \mathbf{E} R(\xi_m)| \\ &\leq \mathbf{E} |R(\omega_m)| + \mathbf{E} |R(\xi_m)| \\ &\lesssim \mathbf{E} \left[\sup_{0 \leq t \leq 1} \|D^3 h(t\omega_m)\| \right] + \mathbf{E} \left[|\xi_m|^3 \sup_{0 \leq t \leq 1} \|D^3 h(t\xi_m)\| \right], \end{aligned}$$

so it remains to bound the third derivative of h .

Recall that

$$h(z) = \frac{1}{2} \sum_s \Phi_\delta(u_s(z), w_s(z)),$$

where $u_s(z) = ((\widehat{f}(s))(z), (\widehat{f}(s+i))(z))$ and $w_s(z) = ((\widehat{f}q(s))(z), (\widehat{f}q(s+i))(z))$. For each s , the dependence of u_s and w_s on the m th coordinate is affine, so we may make the definitions

$$u_s(z) =: u_s^{[m]} + z a_{m,s}, \quad w_s(z) =: w_s^{[m]} + z b_{m,s},$$

where

$$a_{m,s} = \frac{\sqrt{\mu(\pi(m))}}{\sqrt{N}} (\chi_s(\pi(m)), \chi_{s+i}(\pi(m))), \quad b_{m,s} = q(\pi(m)) a_{m,s}.$$

In particular,

$$\|a_{m,s}\|_2 \asymp \sqrt{\mu(\pi(m))/N}, \quad \|b_{m,s}\|_2 \asymp \sqrt{\mu(\pi(m))/N} |q(\pi(m))|.$$

With this notation, the chain rule and the bounds on the third Fréchet derivatives of Φ_δ in [Proposition 8](#) give

$$\begin{aligned} \|D^3 h(z)\| &\lesssim \sum_s \left(\|D_u^3 \Phi_\delta(u_s(z), w_s(z))\| \|a_{m,s}\|_2^3 \right. \\ &\quad + \|D_u^2 D_w \Phi_\delta(u_s(z), w_s(z))\| \|a_{m,s}\|_2^2 \|b_{m,s}\|_2 \\ &\quad \left. + \|D_u D_w^2 \Phi_\delta(u_s(z), w_s(z))\| \|a_{m,s}\|_2 \|b_{m,s}\|_2^2 \right) \\ &\lesssim \mu(\pi(m))^{3/2} \left[\delta^{-3/2} N^{-3/2} \sum_s \|w_s(z)\|_2^2 + \delta^{-1} |q(\pi(m))| N^{-3/2} \sum_s \|w_s(z)\|_2 \right. \\ &\quad \left. + \delta^{-1/2} |q(\pi(m))|^2 N^{-1/2} \right]. \end{aligned}$$

Write

$$Y^{(m,\zeta)} := (\xi_1, \dots, \xi_{m-1}, \zeta, \omega_{m+1}, \dots, \omega_N),$$

so that $h(\zeta) = A_\delta^{(i)}(Y^{(m,\zeta)})$. Then Parseval gives

$$\sum_s \|w_s(\zeta)\|_2^2 = 2 \sum_{k=1}^N \mu(\pi(k)) |q(\pi(k))|^2 |Y_k^{(m,\zeta)}|^2.$$

For the Steinhaus interpolation segment, since $|\omega_m| = 1$,

$$\sup_{0 \leq t \leq 1} \sum_s \|w_s(t\omega_m)\|_2^2 = 2 \sum_{k < m} \mu(\pi(k)) |q(\pi(k))|^2 |\xi_k|^2 + 2 \sum_{k \geq m} \mu(\pi(k)) |q(\pi(k))|^2.$$

Taking expectations and using $\mathbf{E} |\xi_k|^2 = 1$ gives

$$\mathbf{E} \left[\sup_{0 \leq t \leq 1} \sum_s \|w_s(t\omega_m)\|_2^2 \right] = 2 \sum_k \mu(\pi(k)) |q(\pi(k))|^2 = 2.$$

For the Gaussian interpolation segment,

$$\begin{aligned} \sup_{0 \leq t \leq 1} \sum_s \|w_s(t\xi_m)\|_2^2 &= 2 \sum_{k < m} \mu(\pi(k)) |q(\pi(k))|^2 |\xi_k|^2 + 2\mu(\pi(m)) |q(\pi(m))|^2 |\xi_m|^2 \\ &\quad + 2 \sum_{k > m} \mu(\pi(k)) |q(\pi(k))|^2. \end{aligned}$$

Therefore, by independence and the finiteness of the Gaussian moments,

$$\begin{aligned}
\mathbf{E} \left[|\xi_m|^3 \sup_{0 \leq t \leq 1} \sum_s \|w_s(t\xi_m)\|_2^2 \right] &= 2 \mathbf{E} |\xi_m|^3 \sum_{k < m} \mu(\pi(k)) |q(\pi(k))|^2 \mathbf{E} |\xi_k|^2 \\
&\quad + 2 \mu(\pi(m)) |q(\pi(m))|^2 \mathbf{E} |\xi_m|^5 \\
&\quad + 2 \mathbf{E} |\xi_m|^3 \sum_{k > m} \mu(\pi(k)) |q(\pi(k))|^2 \\
&\lesssim \sum_k \mu(\pi(k)) |q(\pi(k))|^2 \\
&= 1.
\end{aligned}$$

Also, by Cauchy–Schwarz,

$$\sum_s \|w_s(z)\|_2 \leq \sqrt{N} \left(\sum_s \|w_s(z)\|_2^2 \right)^{1/2}.$$

Consequently,

$$\mathbf{E} \left[\sup_{0 \leq t \leq 1} \sum_s \|w_s(t\omega_m)\|_2 \right] \leq \sqrt{N} \left(\mathbf{E} \left[\sup_{0 \leq t \leq 1} \sum_s \|w_s(t\omega_m)\|_2^2 \right] \right)^{1/2} \lesssim \sqrt{N},$$

and

$$\begin{aligned}
\mathbf{E} \left[|\xi_m|^3 \sup_{0 \leq t \leq 1} \sum_s \|w_s(t\xi_m)\|_2 \right] &\leq \sqrt{N} (\mathbf{E} |\xi_m|^3)^{1/2} \left(\mathbf{E} \left[|\xi_m|^3 \sup_{0 \leq t \leq 1} \sum_s \|w_s(t\xi_m)\|_2^2 \right] \right)^{1/2} \\
&\lesssim \sqrt{N}.
\end{aligned}$$

Substituting $\delta = \eta/N$, we conclude that

$$\left| \mathbf{E}[h(\omega_m) - h(\xi_m)] \right| \lesssim_\eta \mu(\pi(m))^{3/2} (1 + |q(\pi(m))|^2).$$

Summing over m in the Lindeberg telescoping identity yields

$$\left| \mathbf{E}_{f \sim F_\mu} A_\delta^{(i)} - \mathbf{E}_{\text{Gauss}(\mu)} A_\delta^{(i)} \right| \lesssim_\eta \sum_x \mu(x)^{3/2} (1 + |q(x)|^2).$$

Finally,

$$\sum_x \mu(x)^{3/2} \leq \|\mu\|_\infty^{1/2}, \quad \sum_x \mu(x)^{3/2} |q(x)|^2 \leq \|\mu\|_\infty^{1/2} \sum_x \mu(x) |q(x)|^2 = \|\mu\|_\infty^{1/2},$$

so

$$\left| \mathbf{E}_{f \sim F_\mu} A_\delta^{(i)} - \mathbf{E}_{\text{Gauss}(\mu)} A_\delta^{(i)} \right| \lesssim_\eta \|\mu\|_\infty^{1/2}.$$

This proves the proposition. □

5.2.3 Bounding $\mathbf{E} B_\delta^{(i)}$

This subsection is devoted to the following:

Proposition 10. *For any $p \in (0, 1)$ and with $\delta = \eta/N$,*

$$\mathbf{E}_{f \sim F_\mu} B_\delta \lesssim \eta^{p/4}.$$

Proof. To begin, separate the w 's and the u 's with Cauchy-Schwarz:

$$B_\delta = \sum_s \|w_s\|_2^2 \cdot \frac{\delta}{\|u_s\|_2^2 + \delta} \leq \left(\sum_s \|w_s\|_2^4 \right)^{1/2} \left(\sum_s \left(\frac{\delta}{\|u_s\|_2^2 + \delta} \right)^2 \right)^{1/2}$$

so that (after another Cauchy-Schwarz),

$$\begin{aligned} \mathbf{E} B_\delta &\leq \left(\mathbf{E} \sum_s \|w_s\|_2^4 \right)^{1/2} \left(\mathbf{E} \sum_s \left(\frac{\delta}{\|u_s\|_2^2 + \delta} \right)^2 \right)^{1/2} \\ &\leq \left(\mathbf{E} \sum_s \|w_s\|_2^4 \right)^{1/2} \left(\mathbf{E} \sum_s \frac{\delta}{\|u_s\|_2^2 + \delta} \right)^{1/2} \end{aligned}$$

The first term is controlled again by [Lemma 2](#): $\mathbf{E} \sum_s \|w_s\|_2^4 \lesssim 1/N$. For the second term, note that for all $y > 0$ and $0 < p < 2$,

$$\frac{1}{1+y} \leq y^{-p/2}.$$

Thus,

$$\begin{aligned} \mathbf{E} \frac{\delta}{\|u_s\|_2^2 + \delta} &= \mathbf{E} \frac{\eta}{N\|u_s\|_2^2 + \eta} \\ &\leq \eta^{p/2} \mathbf{E} \left[(N\|u_s\|_2^2)^{-p/2} \right] \\ &\leq \eta^{p/2} \mathbf{E} \left[(N|\widehat{f}(s)|^2)^{-p/2} \right] \\ &= \left(\frac{\eta}{N} \right)^{p/2} \mathbf{E} \left[|\widehat{f}(s)|^{-p} \right], \end{aligned}$$

where we have used $\|u_s\|_2^2 = |\widehat{f}(s)|^2 + |\widehat{f}(s+i)|^2 \geq |\widehat{f}(s)|^2$. Now by the negative-moment Khintchine inequality for Steinhaus sums [\[RTW25\]](#), we have for exponent $q = -p \in (-1, 0)$ that,

$$\left(\mathbf{E} |\widehat{f}(s)|^q \right)^{1/q} \geq C_q \left(\mathbf{E} |\widehat{f}(s)|^2 \right)^{1/2} = C_q \left(\frac{1}{N} \mathbf{E} \sum_x \mu(x) \right)^{1/2} = \frac{C_q}{\sqrt{N}}$$

where $C_q > 0$ is an absolute constant depending on q only. Therefore $\mathbf{E} |\widehat{f}(s)|^{-p} \leq C_{-p}^{-p} N^{p/2}$ and so

$$\mathbf{E} \frac{\delta}{\|u_s\|_2^2 + \delta} \lesssim_p \eta^{p/2}$$

Combining these bounds, we conclude:

$$\mathbf{E} B_\delta^{(i)} \lesssim_p \sqrt{\frac{1}{N}} \sqrt{N \eta^{p/2}} = \eta^{p/4}. \quad \square$$

5.2.4 Putting it together

Proof of [Lemma 3](#). Fix $t > 0$. Taking $p = 1/2$ in [Proposition 10](#), there is a universal constant $C > 0$ such that, whenever $\delta = \eta/N$,

$$\mathbf{E}_{f \sim F_\mu} B_\delta^{(i)} \leq C \eta^{1/8}.$$

Choose $\eta \in (0, 1]$ sufficiently small that

$$C \eta^{1/8} \leq \frac{t}{3},$$

and set

$$\delta := \frac{\eta}{N}.$$

On the event where [Lemma 4](#) holds and

$$\|\mu\|_\infty = N^{-1+o(1)},$$

the following estimates hold simultaneously for every μ -compatible q :

$$\begin{aligned} \mathbf{E}_{f \sim F_\mu} \mathbf{Avg}_{\widehat{f}}[\widehat{fq}] &= \mathbf{E}_{f \sim F_\mu} \left[\frac{1}{n} \sum_i A^{(i)} \right] \\ &\leq \mathbf{E}_{f \sim F_\mu} \left[\frac{1}{n} \sum_i \left(A_\delta^{(i)} + B_\delta^{(i)} \right) \right] \\ &\leq \mathbf{E}_{f \sim \text{Gauss}(\mu)} \left[\frac{1}{n} \sum_i A_\delta^{(i)} \right] + \frac{1}{n} \sum_i \left| \mathbf{E}_{f \sim F_\mu} A_\delta^{(i)} - \mathbf{E}_{f \sim \text{Gauss}(\mu)} A_\delta^{(i)} \right| + \frac{1}{n} \sum_i \mathbf{E}_{f \sim F_\mu} B_\delta^{(i)} \\ &\leq \mathbf{E}_{f \sim \text{Gauss}(\mu)} \left[\frac{1}{n} \sum_i A^{(i)} \right] + C_\eta \|\mu\|_\infty^{1/2} + C\eta^{1/8} \\ &\leq \frac{1}{2} + \mathcal{O}\left(\frac{1}{n}\right) + C_\eta \|\mu\|_\infty^{1/2} + C\eta^{1/8}. \end{aligned}$$

Here the first inequality uses that $A^{(i)} \leq A_\delta^{(i)} + B_\delta^{(i)}$ and the third uses that $A_\delta^{(i)} \leq A^{(i)}$ pointwise in the Gaussian model.

The parameter η is now fixed. Therefore, for all sufficiently large n ,

$$\mathcal{O}\left(\frac{1}{n}\right) + C_\eta \|\mu\|_\infty^{1/2} \leq \frac{2t}{3}.$$

Together with the choice of η , this gives

$$\mathbf{E}_{f \sim F_\mu} \mathbf{Avg}_{\widehat{f}}[\widehat{fq}] \leq \frac{1}{2} + t.$$

The event used above has probability $1 - \exp(-\Omega(n))$ over $\mu \sim \boldsymbol{\mu}$, which proves the lemma. $\square$

5.3 Concentrating $\mathbf{Avg}_{\widehat{f}}[\widehat{fq}]$

Now that we have shown $\mathbf{E}_{f \sim F_\mu} \mathbf{Avg}_{\widehat{f}}[\widehat{fq}] \leq 1/2 + o(1)$, we turn to concentration. We will show doubly-exponential concentration (in n) for each $A^{(i)}$ individually, and then a union bound over coordinates to finish the argument.

Recall our goal is to get an $\exp(-cN^\alpha)$ tail bound on the deviation of the following quantity from its mean

$$A^{(i)} = \sum_s \frac{|\langle u_s, w_s \rangle|^2}{\|u_s\|_2^2}.$$

As mentioned above, $A^{(i)}$ seems to frustrate naive approaches because it does not have a nice gradient.

Theorem 6. *There exists a universal $c > 0$ such that*

$$\Pr[|A^{(i)} - \mathbf{E} A^{(i)}| > t] \leq \exp(-ct^6 N^{1/3})$$

As before, we work with the following smoothed quantity, which has well-behaved derivatives:

$$A_\delta^{(i)} := \frac{1}{2} \sum_s \frac{|\langle u_s, w_s \rangle|^2}{\|u_s\|_2^2 + \delta}$$

Put $D_\delta = A - A_\delta$. Now D_δ is not amenable to Herbst, but we make the observation that pointwise,

$$D_\delta = \frac{1}{2} \sum_s |\langle u_s, w_s \rangle|^2 \left(\frac{1}{\|u_s\|_2^2} - \frac{1}{\|u_s\|_2^2 + \delta} \right) \stackrel{\text{(Cauchy-Schwarz)}}{\leq} \frac{1}{2} \sum_s \frac{\delta \|w_s\|_2^2}{\|u_s\|_2^2 + \delta} =: B_\delta,$$

and the gradient of B_δ is much friendlier.

Finally, note that from our definitions so far,

$$|A - \mathbf{E} A| \leq |A_\delta - \mathbf{E} A_\delta| + |B_\delta - \mathbf{E} B_\delta| + 2 \mathbf{E} B_\delta.$$

So we will show each of these terms are concentrated in turn.

5.3.1 Concentrating $A_\delta^{(i)}$

A Herbst argument. We begin with the relevant carré du champ, where the gradient is taken with respect to the independent phase coordinates $(\theta_x)_x$.

Proposition 11 (Carré du champ bound). *For every $i \in [n]$,*

$$\Gamma(A_\delta^{(i)}) := \left\| \nabla A_\delta^{(i)} \right\|_2^2 = \sum_x \left| \partial_{\theta_x} A_\delta^{(i)} \right|^2 \lesssim \frac{\|\mu\|_\infty}{\delta} \sum_s \|w_s\|_2^4 + \|\mu|q|^2\|_\infty. \quad (12)$$

Proof. Define

$$\Phi(u_s, w_s) := \frac{|\langle u_s, w_s \rangle|^2}{\|u_s\|_2^2 + \delta},$$

so that

$$A_\delta^{(i)} = \frac{1}{2} \sum_s \Phi(u_s, w_s).$$

By the multivariate chain rule,

$$\partial_{\theta_x} \Phi(u_s, w_s) = \Re \langle \nabla_u^{\mathbb{C}} \Phi(u_s, w_s), \partial_{\theta_x} u_s \rangle + \Re \langle \nabla_w^{\mathbb{C}} \Phi(u_s, w_s), \partial_{\theta_x} w_s \rangle.$$

Here $\nabla_u^{\mathbb{C}}$ and $\nabla_w^{\mathbb{C}}$ denote the usual complex gradients, equivalently $2\partial_{\bar{u}}$ and $2\partial_{\bar{w}}$.

By direct calculation,

$$\begin{aligned} \nabla_u^{\mathbb{C}} \Phi(u_s, w_s) &= \frac{2\overline{\langle u_s, w_s \rangle}}{\|u_s\|_2^2 + \delta} w_s - \frac{2|\langle u_s, w_s \rangle|^2}{(\|u_s\|_2^2 + \delta)^2} u_s, \\ \nabla_w^{\mathbb{C}} \Phi(u_s, w_s) &= \frac{2\langle u_s, w_s \rangle}{\|u_s\|_2^2 + \delta} u_s. \end{aligned}$$

Consequently,

$$\left\| \nabla_u^{\mathbb{C}} \Phi(u_s, w_s) \right\|_2 \lesssim \frac{\|w_s\|_2^2}{\sqrt{\delta}}, \quad \left\| \nabla_w^{\mathbb{C}} \Phi(u_s, w_s) \right\|_2 \lesssim \|w_s\|_2. \quad (13)$$

Write

$$\chi_s(x) := (-1)^{s \cdot x}.$$

Since

$$\hat{f}(s) = \frac{1}{\sqrt{N}} \sum_y \sqrt{\mu(y)} e^{i\theta_y} \chi_s(y)$$

and $g = fq$, we have

$$\partial_{\theta_x} \widehat{f}(s) = i \sqrt{\frac{\mu(x)}{N}} e^{i\theta_x} \chi_s(x)$$

and

$$\partial_{\theta_x} \widehat{g}(s) = q(x) \partial_{\theta_x} \widehat{f}(s).$$

It follows that

$$\partial_{\theta_x} u_s = i \sqrt{\frac{\mu(x)}{N}} e^{i\theta_x} \chi_s(x) (1, (-1)^{x_i})$$

and

$$\partial_{\theta_x} w_s = q(x) \partial_{\theta_x} u_s.$$

Substituting these identities into the chain rule and factoring out the common scalar of modulus $\sqrt{\mu(x)/N}$ gives

$$\begin{aligned} \left| \partial_{\theta_x} A_\delta^{(i)} \right| &\leq \frac{1}{2} \sqrt{\frac{\mu(x)}{N}} \left| \sum_s \chi_s(x) \langle \nabla_u^{\mathbb{C}} \Phi(u_s, w_s), (1, (-1)^{x_i}) \rangle \right| \\ &\quad + \frac{1}{2} \sqrt{\frac{\mu(x)}{N}} |q(x)| \left| \sum_s \chi_s(x) \langle \nabla_w^{\mathbb{C}} \Phi(u_s, w_s), (1, (-1)^{x_i}) \rangle \right|. \end{aligned} \quad (14)$$

For any scalar family $(a_s)_s$ and every $b \in \{0, 1\}$, Parseval on the slice $\{x : x_i = b\}$ gives

$$\sum_{x: x_i=b} \left| \sum_s \chi_s(x) a_s \right|^2 = \frac{N}{2} \sum_{s: s_i=0} |a_s + (-1)^b a_{s+i}|^2 \leq N \sum_s |a_s|^2. \quad (15)$$

Applying (15) on the two slices $x_i = 0$ and $x_i = 1$, and using $\|(1, (-1)^{x_i})\|_2^2 = 2$, we obtain

$$\sum_x \left| \sum_s \chi_s(x) \langle \nabla_u^{\mathbb{C}} \Phi(u_s, w_s), (1, (-1)^{x_i}) \rangle \right|^2 \leq 4N \sum_s \|\nabla_u^{\mathbb{C}} \Phi(u_s, w_s)\|_2^2 \quad (16)$$

and likewise

$$\sum_x \left| \sum_s \chi_s(x) \langle \nabla_w^{\mathbb{C}} \Phi(u_s, w_s), (1, (-1)^{x_i}) \rangle \right|^2 \leq 4N \sum_s \|\nabla_w^{\mathbb{C}} \Phi(u_s, w_s)\|_2^2. \quad (17)$$

Squaring (14), summing over x , and using (16) and (17), we conclude that

$$\left\| \nabla A_\delta^{(i)} \right\|_2^2 \lesssim \|\mu\|_\infty \sum_s \|\nabla_u^{\mathbb{C}} \Phi(u_s, w_s)\|_2^2 + \|\mu|q|^2\|_\infty \sum_s \|\nabla_w^{\mathbb{C}} \Phi(u_s, w_s)\|_2^2.$$

Using (13), this becomes

$$\left\| \nabla A_\delta^{(i)} \right\|_2^2 \lesssim \frac{\|\mu\|_\infty}{\delta} \sum_s \|w_s\|_2^4 + \|\mu|q|^2\|_\infty \sum_s \|w_s\|_2^2.$$

Finally, Parseval and μ -compatibility give

$$\begin{aligned} \sum_s \|w_s\|_2^2 &= \sum_s (|\widehat{g}(s)|^2 + |\widehat{g}(s+i)|^2) \\ &= 2\|g\|_2^2 \\ &= 2 \sum_x \mu(x) |q(x)|^2 \\ &= 2. \end{aligned}$$

Absorbing this factor into the implicit constant proves (12). $\square$

Now δ will be chosen like $\delta = \eta/N$, so we need $W := \sum_s \|w_s\|_2^4 \lesssim \frac{1}{N}$ with high probability. Fortunately, W is nothing but a Steinhaus chaos of fixed degree, and its concentration is given by [Lemma 2](#). We shall combine [Proposition 11](#) and [Lemma 2](#) using the following localized form of the Herbst argument.

Fact 4 (Localized Herbst bound for Steinhaus variables). *There exists a universal constant $c > 0$ such that the following holds. Let F be a centered smooth function of independent Steinhaus variables, and let G be an event such that*

$$|F| \leq 1, \quad \|\nabla F\|_2^2 \leq b \quad \text{on } G, \quad \text{and} \quad \|\nabla F\|_2^2 \leq B \quad \text{everywhere.}$$

If

$$B \Pr(G^c) < b,$$

then, for every

$$0 < t \leq cb \log\left(\frac{b}{B \Pr(G^c)}\right),$$

one has

$$\Pr[|F| > t] \leq 2 \exp\left(-\frac{ct^2}{b}\right).$$

Proof. For $\lambda \geq 0$, write

$$M(\lambda) := \mathbf{E} e^{\lambda F}, \quad \psi(\lambda) := \log M(\lambda).$$

The product Steinhaus log-Sobolev inequality [[Wei80](#)] and the standard Herbst calculation (see, e.g., [[Led99](#)]) give

$$\left(\frac{\psi(\lambda)}{\lambda}\right)' \leq C \frac{\mathbf{E}[e^{\lambda F} \|\nabla F\|_2^2]}{\mathbf{E} e^{\lambda F}}.$$

Since F is centered, Jensen's inequality gives $\mathbf{E} e^{\lambda F} \geq 1$. Since $|F| \leq 1$,

$$\frac{\mathbf{E}[e^{\lambda F} \mathbf{1}_{G^c}]}{\mathbf{E} e^{\lambda F}} \leq e^\lambda \Pr(G^c).$$

It follows that

$$\left(\frac{\psi(\lambda)}{\lambda}\right)' \leq C(b + B \Pr(G^c) e^\lambda).$$

Integrating from 0 to λ , and using

$$e^\lambda - 1 \leq \lambda e^\lambda,$$

gives

$$\psi(\lambda) \leq C\lambda^2(b + B \Pr(G^c) e^\lambda).$$

Therefore, whenever

$$0 \leq \lambda \leq \log\left(\frac{b}{B \Pr(G^c)}\right),$$

we have

$$\psi(\lambda) \leq Cb\lambda^2.$$

Chernoff's bound, with $\lambda = ct/b$ for a sufficiently small universal constant $c > 0$, now gives

$$\Pr[F > t] \leq \exp\left(-\frac{ct^2}{b}\right)$$

throughout the asserted range of t . Applying the same argument to $-F$ proves the result. $\square$

Corollary 3. *There exist universal constants $c, C > 0$ such that the following holds. Let q be μ -compatible, let $\eta \in (0, 1]$, and set*

$$\delta = \frac{\eta}{N}.$$

Suppose that

$$\left(\frac{\eta}{\|\mu\|_\infty} \right)^{1/3} \geq C \log N.$$

Then, for every $i \in [n]$ and every $0 < t \leq 1$,

$$\Pr_{f \sim F_\mu} \left[\left| A_\delta^{(i)} - \mathbf{E}_{f \sim F_\mu} A_\delta^{(i)} \right| > t \right] \leq 2 \exp \left(- \frac{ct^2}{(\|\mu\|_\infty/\eta)^{1/3} + \|\mu|q|^2\|_\infty} \right). \quad (18)$$

Proof. Put

$$F := A_\delta^{(i)} - \mathbf{E}_{f \sim F_\mu} A_\delta^{(i)}.$$

Since

$$0 \leq A_\delta^{(i)} \leq \frac{1}{2} \sum_s \|w_s\|_2^2 = 1,$$

we have

$$|F| \leq 1.$$

Let

$$W := \sum_s \|w_s\|_2^4.$$

Let $K \geq 1$ be a universal constant to be fixed below, and define

$$G := \left\{ W \leq \frac{8}{N} + \frac{K}{N} \left(\frac{\eta}{\|\mu\|_\infty} \right)^{2/3} \right\}. \quad (19)$$

Since $\mathbf{E} W \leq 8/N$, the hypothesis of the corollary ensures, after increasing C if necessary, that the deviation in (19) lies in the range of [Lemma 2](#). Consequently,

$$\Pr_{f \sim F_\mu} [G^c] \leq \exp \left(-c\sqrt{K} \left(\frac{\eta}{\|\mu\|_\infty} \right)^{1/3} \right). \quad (20)$$

On G , [Proposition 11](#) gives

$$\begin{aligned} \left\| \nabla A_\delta^{(i)} \right\|_2^2 &\lesssim \frac{N\|\mu\|_\infty}{\eta} \left[\frac{8}{N} + \frac{K}{N} \left(\frac{\eta}{\|\mu\|_\infty} \right)^{2/3} \right] + \|\mu|q|^2\|_\infty \\ &= 8 \frac{\|\mu\|_\infty}{\eta} + K \left(\frac{\|\mu\|_\infty}{\eta} \right)^{1/3} + \|\mu|q|^2\|_\infty \\ &\lesssim (K+8) \left(\frac{\|\mu\|_\infty}{\eta} \right)^{1/3} + \|\mu|q|^2\|_\infty. \end{aligned} \quad (21)$$

In the last line we used

$$\frac{\|\mu\|_\infty}{\eta} \leq 1,$$

which follows from the hypothesis of the corollary.

We also need the everywhere bound. By Parseval, $\sum_s \|w_s\|_2^2 = 2$, and hence

$$W \leq \left(\sum_s \|w_s\|_2^2 \right)^2 = 4.$$

Therefore, everywhere,

$$\left\| \nabla A_\delta^{(i)} \right\|_2^2 \lesssim \frac{4N \|\mu\|_\infty}{\eta} + \|\mu|q|^2\|_\infty. \quad (22)$$

By (21) and (22), there is a universal constant $C_0 > 0$ such that

$$\|\nabla F\|_2^2 \leq b \quad \text{on } G$$

and

$$\|\nabla F\|_2^2 \leq B \quad \text{everywhere,}$$

where

$$b := C_0 \left[(K+8) \left(\frac{\|\mu\|_\infty}{\eta} \right)^{1/3} + \|\mu|q|^2\|_\infty \right],$$

$$B := C_0 \left[\frac{4N \|\mu\|_\infty}{\eta} + \|\mu|q|^2\|_\infty \right].$$

Because q is μ -compatible,

$$\|\mu|q|^2\|_\infty \leq \sum_x \mu(x) |q(x)|^2 = 1.$$

Using also $\|\mu\|_\infty/\eta \leq 1$, we obtain

$$\frac{B}{b} \leq C \left[N \left(\frac{\|\mu\|_\infty}{\eta} \right)^{2/3} + 1 \right] \leq CN.$$

Combining this with (20) gives

$$\log \left(\frac{b}{B \Pr_{f \sim F_\mu}(G^c)} \right) \geq c\sqrt{K} \left(\frac{\eta}{\|\mu\|_\infty} \right)^{1/3} - C \log N$$

$$\geq \frac{c\sqrt{K}}{2} \left(\frac{\eta}{\|\mu\|_\infty} \right)^{1/3},$$

after increasing the constant in the hypothesis of the corollary. In particular,

$$B \Pr_{f \sim F_\mu}(G^c) < b.$$

Moreover,

$$b \geq C_0 K \left(\frac{\|\mu\|_\infty}{\eta} \right)^{1/3},$$

and therefore

$$cb \log \left(\frac{b}{B \Pr_{f \sim F_\mu}(G^c)} \right) \geq cK^{3/2}.$$

We now fix the universal constant K large enough that the right-hand side is at least 1. Thus the admissible range in [Fact 4](#) contains every $t \in (0, 1]$.

Applying [Fact 4](#), we obtain

$$\Pr_{f \sim F_\mu} [|F| > t] \leq 2 \exp\left(-\frac{ct^2}{b}\right).$$

Since K is now a fixed universal constant,

$$b \lesssim \left(\frac{\|\mu\|_\infty}{\eta}\right)^{1/3} + \|\mu|q|^2\|_\infty,$$

which proves [\(18\)](#). □

5.3.2 Concentrating $B_\delta^{(i)}$

The same carré-du-champ estimate holds for the error dominator.

Corollary 4. *For every $i \in [n]$,*

$$\left\|\nabla B_\delta^{(i)}\right\|_2^2 \lesssim \frac{\|\mu\|_\infty}{\delta} \sum_s \|w_s\|_2^4 + \|\mu|q|^2\|_\infty.$$

Proof. Define

$$\Phi(u_s, w_s) := \frac{\delta \|w_s\|_2^2}{\|u_s\|_2^2 + \delta},$$

so that

$$B_\delta^{(i)} = \frac{1}{2} \sum_s \Phi(u_s, w_s).$$

By direct calculation,

$$\nabla_u^{\mathbb{C}} \Phi(u_s, w_s) = -\frac{2\delta \|w_s\|_2^2}{(\|u_s\|_2^2 + \delta)^2} u_s$$

and

$$\nabla_w^{\mathbb{C}} \Phi(u_s, w_s) = \frac{2\delta}{\|u_s\|_2^2 + \delta} w_s.$$

Hence

$$\left\|\nabla_u^{\mathbb{C}} \Phi(u_s, w_s)\right\|_2 \lesssim \frac{\|w_s\|_2^2}{\sqrt{\delta}}, \quad \left\|\nabla_w^{\mathbb{C}} \Phi(u_s, w_s)\right\|_2 \lesssim \|w_s\|_2.$$

The phase derivatives of u_s and w_s are unchanged, so the Parseval argument from [Proposition 11](#) applies without further change. □

We therefore obtain the same concentration estimate as for $A_\delta^{(i)}$.

Corollary 5 (Concentration of the error dominator). *There exist universal constants $c, C > 0$ such that the following holds. Let q be μ -compatible, let $\eta \in (0, 1]$, and set*

$$\delta = \frac{\eta}{N}.$$

Suppose that

$$\left(\frac{\eta}{\|\mu\|_\infty}\right)^{1/3} \geq C \log N.$$

Then, for every $i \in [n]$ and every $0 < t \leq 1$,

$$\Pr_{f \sim F_\mu} \left[\left| B_\delta^{(i)} - \mathbf{E}_{f \sim F_\mu} B_\delta^{(i)} \right| > t \right] \leq 2 \exp\left(-\frac{ct^2}{(\|\mu\|_\infty/\eta)^{1/3} + \|\mu|q|^2\|_\infty}\right). \quad (23)$$

Proof. Put

$$F := B_\delta^{(i)} - \mathbf{E}_{f \sim F_\mu} B_\delta^{(i)}.$$

Since

$$0 \leq B_\delta^{(i)} \leq \frac{1}{2} \sum_s \|w_s\|_2^2 = 1,$$

we have

$$|F| \leq 1.$$

Let

$$W := \sum_s \|w_s\|_2^4.$$

Choose the same sufficiently large universal constant K as in the proof of [Corollary 3](#), and define

$$G := \left\{ W \leq \frac{8}{N} + \frac{K}{N} \left(\frac{\eta}{\|\mu\|_\infty} \right)^{2/3} \right\}.$$

As in that proof, [Lemma 2](#) gives

$$\Pr_{f \sim F_\mu} [G^c] \leq \exp \left(-c\sqrt{K} \left(\frac{\eta}{\|\mu\|_\infty} \right)^{1/3} \right).$$

On G , [Corollary 4](#) gives

$$\left\| \nabla B_\delta^{(i)} \right\|_2^2 \lesssim (K+8) \left(\frac{\|\mu\|_\infty}{\eta} \right)^{1/3} + \|\mu|q|^2\|_\infty.$$

Moreover, since

$$W \leq \left(\sum_s \|w_s\|_2^2 \right)^2 = 4,$$

we have the global bound

$$\left\| \nabla B_\delta^{(i)} \right\|_2^2 \lesssim \frac{4N\|\mu\|_\infty}{\eta} + \|\mu|q|^2\|_\infty.$$

These are exactly the good-event gradient bound, global gradient bound, and exceptional-event estimate used in the proof of [Corollary 3](#). Together with $|F| \leq 1$, they verify the hypotheses of [Fact 4](#) with the same choices of b , B , and K . Applying that fact proves (23). $\square$

5.3.3 Combining the bounds

We now remove the smoothing and deduce concentration of $A^{(i)}$.

Proposition 12 (Per-coordinate concentration for flat q). *There exist universal constants $c, C > 0$ such that the following holds. Let $t \in (0, 1]$, let q be μ -compatible, and suppose that*

$$\|\mu|q|^2\|_\infty \leq \|\mu\|_\infty^{1/3}$$

and

$$\left(\frac{t^8}{\|\mu\|_\infty} \right)^{1/3} \geq C \log N.$$

Then, for every $i \in [n]$,

$$\Pr_{f \sim F_\mu} \left[\left| A^{(i)} - \mathbf{E}_{f \sim F_\mu} A^{(i)} \right| > t \right] \leq 4 \exp \left(-ct^6 \|\mu\|_\infty^{-1/3} \right).$$

Proof. Take $p = 1/2$ in [Proposition 10](#), so that for $\delta = \eta/N$,

$$\mathbf{E}_{f \sim F_\mu} B_\delta^{(i)} \leq C\eta^{1/8}$$

for a universal constant $C > 0$. Choose a sufficiently small universal constant $c_0 > 0$ and set

$$\eta := c_0 t^8, \quad \delta := \frac{\eta}{N},$$

so that

$$2 \mathbf{E}_{f \sim F_\mu} B_\delta^{(i)} \leq \frac{t}{2}.$$

After increasing the constant in the hypothesis of the proposition, we also have

$$\left(\frac{\eta}{\|\mu\|_\infty} \right)^{1/3} \geq C \log N,$$

so [Corollary 3](#) and [Corollary 5](#) apply.

Recall that

$$\left| A^{(i)} - \mathbf{E} A^{(i)} \right| \leq \left| A_\delta^{(i)} - \mathbf{E} A_\delta^{(i)} \right| + \left| B_\delta^{(i)} - \mathbf{E} B_\delta^{(i)} \right| + 2 \mathbf{E} B_\delta^{(i)}.$$

It follows that

$$\begin{aligned} \Pr_{f \sim F_\mu} \left[\left| A^{(i)} - \mathbf{E} A^{(i)} \right| > t \right] &\leq \Pr_{f \sim F_\mu} \left[\left| A_\delta^{(i)} - \mathbf{E} A_\delta^{(i)} \right| > \frac{t}{4} \right] + \Pr_{f \sim F_\mu} \left[\left| B_\delta^{(i)} - \mathbf{E} B_\delta^{(i)} \right| > \frac{t}{4} \right] \\ &\leq 4 \exp \left(- \frac{ct^2}{(\|\mu\|_\infty/\eta)^{1/3} + \|\mu|q|^2\|_\infty} \right). \end{aligned}$$

By the flatness hypothesis and the fact that $\eta \leq 1$,

$$\begin{aligned} \left(\frac{\|\mu\|_\infty}{\eta} \right)^{1/3} + \|\mu|q|^2\|_\infty &\leq \eta^{-1/3} \|\mu\|_\infty^{1/3} + \|\mu\|_\infty^{1/3} \\ &\leq 2\eta^{-1/3} \|\mu\|_\infty^{1/3}. \end{aligned}$$

Therefore,

$$\Pr_{f \sim F_\mu} \left[\left| A^{(i)} - \mathbf{E} A^{(i)} \right| > t \right] \leq 4 \exp \left(-ct^2 \eta^{1/3} \|\mu\|_\infty^{-1/3} \right).$$

Since $\eta = c_0 t^8$, this gives

$$\Pr_{f \sim F_\mu} \left[\left| A^{(i)} - \mathbf{E} A^{(i)} \right| > t \right] \leq 4 \exp \left(-ct^{14/3} \|\mu\|_\infty^{-1/3} \right).$$

Finally, because $0 < t \leq 1$,

$$t^{14/3} \geq t^6,$$

which proves the proposition. $\square$

5.4 Finishing up

Proof of [Theorem 5](#). Fix $t > 0$. It suffices to consider $t \in (0, 1]$, since

$$0 \leq \mathbf{Avg}_{\widehat{f}}[\widehat{f}q] \leq 1.$$

With probability $1 - \exp(-\Omega(n))$ over $\mu \sim \boldsymbol{\mu}$, simultaneously for every μ -compatible q ,

$$\frac{1}{n} \sum_i \mathbf{E}_{f \sim F_\mu} A^{(i)} \leq \frac{1}{2} + o(1)$$

by [Lemma 3](#), and

$$\|\mu\|_\infty = N^{-1+o(1)}.$$

Fix such a μ and a μ -compatible q satisfying

$$\|\mu|q|^2\|_\infty \leq \|\mu\|_\infty^{1/3}.$$

For fixed $t > 0$ and all sufficiently large n ,

$$\left(\frac{t^8}{\|\mu\|_\infty} \right)^{1/3} \geq C \log N.$$

Thus [Proposition 12](#) gives, for every $i \in [n]$,

$$\Pr_{f \sim F_\mu} \left[A^{(i)} - \mathbf{E}_{f \sim F_\mu} A^{(i)} > t \right] \leq 4 \exp(-ct^6 N^{1/3-o(1)}).$$

Recalling that $\mathbf{Avg}_{\widehat{f}}[\widehat{f}q] = \frac{1}{n} \sum_i A^{(i)}$, we obtain

$$\begin{aligned} & \Pr_{f \sim F_\mu} \left[\mathbf{Avg}_{\widehat{f}}[\widehat{f}q] \geq \frac{1}{2} + t + o(1) \right] \\ & \leq \Pr_{f \sim F_\mu} \left[\frac{1}{n} \sum_i \left(A^{(i)} - \mathbf{E}_{f \sim F_\mu} A^{(i)} \right) \geq t \right] \\ & \leq \sum_i \Pr_{f \sim F_\mu} \left[A^{(i)} - \mathbf{E}_{f \sim F_\mu} A^{(i)} \geq t \right] \\ & \leq 4n \exp(-ct^6 N^{1/3-o(1)}) \\ & \leq \exp(-ct^6 N^{1/3-o(1)}), \end{aligned}$$

where the last line follows, after decreasing c , for all sufficiently large n . This proves the theorem. $\square$

Interlude: Proof of the uncertainty principle for phase states

As a warmup to the main theorem, we take a moment to prove the uncertainty principle in the special case of random phase states. The contents of this section are not required for the proof of the main theorem and the reader may freely skip it.

Theorem 7. *For n large enough, for f a random phase function, for all g compatible with f ,*

$$\mathbf{Avg}_f[g] + \mathbf{Avg}_{\widehat{f}}[\widehat{g}] \leq 1.99905,$$

except with probability $2^{-\Omega(n)}$ over the choice of f .

We need to bound the covering number of the set of low-influence functions

$$\mathcal{Q}_{n,\kappa} := \{q : \{0,1\}^n \rightarrow \mathbb{C} : \|q\|_{L^2(\text{unif})}^2 = 1, \mathbf{Inf}[q] \leq \kappa n\}.$$

The following covering number bound is fairly standard but we include it for completeness.

Proposition 13. *The $L^2(\text{unif})$ ε -covering number of $\mathcal{Q}_{n,\kappa}$ bounded as*

$$\mathcal{N}(\varepsilon, \mathcal{Q}_{n,\kappa}) \leq \exp\left(2^{H_2(\min\{\kappa/\varepsilon^2, 1/2\})n} \cdot \log \frac{C}{\varepsilon^2}\right),$$

where H_2 is the binary entropy.

Proof. Define $f_{\leq d} = \sum_{|S| \leq d} \widehat{f}(S) \chi_S$. Then

$$\|f - f_{\leq d}\|_2^2 = \sum_{\substack{S \subseteq [n] \\ |S| > d}} \widehat{f}(S)^2 \leq \frac{1}{d} \sum_{S \subseteq [n]} |S| \widehat{f}(S)^2 = \frac{I[f]}{d} \leq \frac{\kappa n}{d},$$

and choosing $d = \lfloor \kappa n / \varepsilon^2 \rfloor$ gives $\|f - f_{\leq d}\|_2 \leq \varepsilon$.

Thus $\mathcal{Q}_{n,\kappa}$ is ε -covered by the unit ball of the subspace $\text{Span}\{\chi_S : |S| \leq d\}$, whose dimension is

$$D_d = \sum_{j=0}^d \binom{n}{j} \leq 2^{H_2(d/n)n} \leq 2^{H_2(\kappa/\varepsilon^2)n},$$

where the inequalities hold provided $d \leq n/2$. A standard volumetric bound for the Euclidean ball in $\mathbb{C}^{D_d}$ yields

$$\mathcal{N}(\varepsilon, \mathcal{Q}_{n,\kappa}) \leq \exp\left(D_d \log \frac{C}{\varepsilon^2}\right)$$

for a universal constant C . Substituting for D_d yields the quoted bound. $\square$

We are now prepared to prove [Theorem 7](#).

Proof of Theorem 7. Let κ be an influence threshold to be chosen later and let $g =: fq$, which is well-defined with probability 1. If $\mathbf{Inf}[q] \geq \kappa$, then $\mathbf{Avg}_f[q] \leq 1 - \kappa$.

Otherwise, $\mathbf{Inf}[q] \leq \kappa$ and by [Proposition 13](#)

$$\log \mathcal{N}(\varepsilon, \mathcal{Q}_{n,\kappa}) \leq \mathcal{O}\left(2^{H_2(\min\{\kappa/\varepsilon^2, 1/2\})n} \cdot \log \frac{1}{\varepsilon}\right),$$

Let $\text{Net}_\varepsilon(\mathcal{Q}_{n,\kappa})$ be a minimum-cardinality ε -net for $\mathcal{Q}_{n,\kappa}$. Then by [Theorem 5](#), we have after putting $t = \varepsilon$ that

$$\Pr_f \left[\forall q \in \text{Net}_\varepsilon(\mathcal{Q}_{n,\kappa}), \mathbf{Avg}_{\widehat{f}}[\widehat{fq}] \geq \frac{1}{2} + 2\varepsilon + o(1) \right] \leq \exp\left(2^{H_2(\min\{\kappa/\varepsilon^2, 1/2\})n} \cdot \log \frac{C}{\varepsilon^2} - c\varepsilon^6 2^{n/3}\right)$$

Therefore for any $q \in \mathcal{Q}_{n,\kappa}$ and the closest $q^* \in \text{Net}_\varepsilon(\mathcal{Q}_{n,\kappa})$ to q ,

$$\begin{aligned} \mathbf{Avg}_{\widehat{f}}[\widehat{fq}] &=: \langle q, \widetilde{A}q \rangle \leq \langle q^*, \widetilde{A}q^* \rangle + 2\|\widetilde{A}\|_{\text{op}}\|q - q^*\|_{L^2(\text{unif})} \\ &\leq \frac{1}{2} + 4\varepsilon + o(1) \end{aligned}$$

with high probability over f , provided

$$H_2\left(\min\left\{\frac{\kappa}{\varepsilon^2}, \frac{1}{2}\right\}\right) < \frac{1}{3}. \quad (24)$$

Conditioned on this event, we therefore have

$$\mathbf{Avg}_f[g] + \mathbf{Avg}_{\widehat{f}}[\widehat{g}] \leq \max\left\{2 - \kappa, \frac{3}{2} + 4\varepsilon + o(1)\right\}$$

It remains to optimize $\kappa, \varepsilon > 0$ subject to the constraint of (24). One setting close to the optimum is

$$\kappa \approx 0.00096 \quad \varepsilon \approx 0.12476,$$

yielding the explicit upper bound of $\mathbf{Avg}_f[g] + \mathbf{Avg}_{\widehat{f}}[\widehat{g}] \leq 1.99905$. $\square$

6 The A - B partition and comparing Dirichlet forms on A

We now return to the proof of [Theorem 4](#). As outlined in the technical overview, the next step is to split $\{0, 1\}^n$ into two sets, a large one A such that the ε -net argument works on functions supported on A , and a small one B where the ε -net argument is false but we can still bound $\mathbf{Avg}$ for functions supported here by other means. In particular, for a universal constant $T > 1$ to be chosen later, set

$$A = \left\{ x : \mu(x) \in \left[\frac{1}{TN}, \frac{T}{N} \right] \right\} \quad \text{and} \quad B = A^c.$$

Define $q_A = q \cdot \mathbf{1}_A$ and $q_B = q \cdot \mathbf{1}_B$.

The idea is that $\mathbf{Inf}_\mu$ should be comparable (up to T) with $\mathbf{Inf}_{\text{unif}}$ on A , so the covering number facts that hold for low-influence functions on the uniform-measure hypercube should hold also for q_A . Unfortunately, it is not always true that q_A is low influence, even though q is promised to be. Recall that in the win-win argument, we are only given $\mathbf{Inf}_\mu(q) \leq \kappa n$. The only immediate fact we can conclude about q_A from the given bound is that

$$\mathbf{Inf}_{\text{unif}}^{\text{int}(A)}(q_A) \simeq_T \mathbf{Inf}_\mu^{\text{int}(A)}(q_A) = \mathbf{Inf}_\mu^{\text{int}(A)}(q) \stackrel{(\text{inclusion})}{\leq} \mathbf{Inf}_\mu(q) \leq \kappa,$$

where for any function h ,

$$\mathbf{Inf}_\mu^{\text{int}(A)}(h) := \sum_{\substack{x, y \in A \\ y \sim x}} \frac{2\mu(x)\mu(y)}{\mu(x) + \mu(y)} \left| \frac{h(x) - h(y)}{2} \right|^2.$$

Unfortunately, without using more about A , having bounded internal influence $\mathbf{Inf}_\mu^{\text{int}(A)}$ is far from enough to get a covering number bound. A simple counterexample is A that is an independent set on the hypercube graph (no interior); a more general one is A any collection of very many small, disjoint subgraphs A_j , $j = 1, \dots, J$. As long as q is constant on each of the A_j 's, $\mathbf{Inf}_{\text{unif}}^{\text{int}(A)}(q_A)$ will always be zero, even though such functions constitute a J -dimensional subspace.

But these examples merely show we must use more about A . The main purpose of this section is to show that with high probability over μ , the geometry of A is such that a bound $\mathbf{Inf}_{\text{unif}}^{\text{int}(A)}(q_A)$ indeed suffices to control the metric entropy of the q_A 's.

Theorem 8 (Metric entropy of A -part of low-energy q). *Let*

$$Q_\kappa = \left\{ q_A : \mathbf{E}_\mu q = 0, \|q\|_{L^2(\mu)} = 1, \mathbf{Inf}_\mu^{\text{int}(A)}(q) \leq \kappa n \right\}$$

Then with high probability over $\mu \sim \boldsymbol{\mu}$, for all $\varepsilon \in (0, 1]$,

$$\log \mathcal{N}_\varepsilon(Q_\kappa, L^2(\mu)) \leq N^{H(\gamma) + o(1)} \log(C/\varepsilon) \quad \text{for} \quad \gamma := C \frac{T^5 \kappa}{\varepsilon^2}.$$

Here and throughout, $\mathcal{N}_\varepsilon(S, L^2(\mu))$ denotes the ε -covering number of a set S with respect to the metric induced by $L^2(\mu)$.

The main idea in the proof is to construct a *linear extension operator* Ext_A so that $(\text{Ext}_A q_A)_A = q_A$ and

$$\mathbf{Inf}_{\text{unif}}(\text{Ext}_A q_A) \lesssim_T \mathbf{Inf}_{\text{unif}}^{\text{int}(A)}(q_A) \stackrel{(\text{by the above})}{\leq} \kappa n.$$

We may then apply standard dimension-counting argument for low-influence function to the $\text{Ext}_A q_A$'s.

To begin, for any $b \in B$ write $\mathcal{N}_A(b) = \{x \in A : x \sim b\}$. Then define Ext_A via

$$(\text{Ext}_A h)(x) = \begin{cases} h(x), & x \in A, \\ \frac{1}{|\mathcal{N}_A(b)|} \sum_{u \in \mathcal{N}_A(b)} h(u), & x = b \in B. \end{cases}$$

for any $h : \{0, 1\}^n \rightarrow \mathbb{C}$. Clearly, $(\text{Ext}_A h)_A = h_A$. Our analysis of Ext_A draws inspiration from the literature on canonical paths and the “local escape property” of [HPS25]. We will use the canonical paths employed in [HPS25]; let us recall these and then assemble some facts about them.

Definition 2 (Canonical i -paths, adapted from [HPS25]). *Let $(u, v) \in \{0, 1\}^n$. For $i \in [n]$ define the i^{th} canonical path, $P_i(u, v)$ to be as follows.*

- Let $D = \{j : u_j \neq v_j\}$.
- Begin at u and flip coordinate i
- Scan rightward (cyclically) and flip each bit in D .
- If $i \notin D$, flip i again at the end.

Fact 5 (Adapted from [HPS25, Prop. 11]). *For any two vertices $x, y \in \{0, 1\}^n$, the paths $P_i(x, y)$ are simple, pairwise internally disjoint, and of length $|P_i(x, y)| \leq d(x, y) + 2$.*

We need a version of the local escape property of [HPS25] for general A (not just the thresholds $[\frac{1}{11N}, \frac{5}{N}]$).

Proposition 14 (Local escape property, generalized from [HPS25, Lemma 17]). *For any $T \geq 12$ there are $\alpha = \alpha(T), \beta = \beta(T) > 0$ such that for all n large enough, with probability at least $1 - 2^{-\beta n}$,*

- each $x \in \{0, 1\}^n$ has at least αn neighbors in A ;
- for all x, y with $d(x, y) \leq 3$, at least αn of the i -paths $P_i(x, y)$ have all internal vertices in A .

Proof. Let us use a normalized exponential model for μ . That is, define Z_x , $x \in \{0, 1\}^n$ to be iid $\text{Exp}(1)$ and let $\mu(x) = Z_x / (\sum_y Z_y)$. Let

$$E = \{N/2 \leq \sum_x Z_x \leq 2N\}$$

and as usual, $\Pr[E] \geq 1 - \exp(-\Omega(N))$. Let us define a “surrogate” A via

$$\tilde{A} = \{x : 2/T \leq Z_x \leq T/2\}$$

and note that if E obtains, $\tilde{A} \subseteq A$, because

$$\frac{1}{TN} \leq \mu(x) = \frac{Z_x}{\sum_y Z_y} \leq \frac{T}{N}.$$

Also note for any fixed x ,

$$p_{\tilde{A}} := \Pr[x \in \tilde{A}] = e^{-2/T} - e^{-T/2},$$

so for all $T \geq 12$,

$$p_{\tilde{A}}^4 > \frac{1}{2}.$$

Proof of (i). Denote the number of $\tilde{A}$ -neighbors of x by $G_x = |\{j \in [n] : x + e_j \in \tilde{A}\}|$. By independence, $G_x \sim \text{Bin}(n, p_{\tilde{A}})$ and so for the appropriate choice of α_1 ,

$$\Pr[G_x < \alpha_1 n] \leq e^{-c_1 n}$$

for $c_1 > \log 2$. (This is possible because $p_{\tilde{A}} > 1/2$.) Union-bounding over x gives

$$\Pr[\exists x : G_x < \alpha_1 n] \leq 1 - 2^n e^{-c_1 n} \leq 2^{\beta_1 n}$$

for an appropriate choice of β_1 and all n large enough.

Proof of (ii). Fix $x, y \in \{0, 1\}^n$ with $d(x, y) \leq 3$. By [Fact 5](#) the paths $P_i(x, y)$, $i \in [n]$ are internally disjoint and have length at most $d(x, y) + 2 \leq 5$. So each path has at most 4 internal vertices, disjoint from all the others. Let $J_i(x, y)$ indicate the event that all internal vertices of $P_i(x, y)$ lie in $\tilde{A}$. By disjointness, the J_i 's are independent and

$$\Pr[J_i(x, y) = 1] \geq q_{\tilde{A}}^4 > \frac{1}{2}.$$

Thus, with $K(x, y) := \sum_i J_i(x, y)$, we may choose an α_2 so that

$$\Pr[K(x, y) < \alpha_2 n] \leq e^{-c_2 n}$$

where again $c_2 > \log 2$.

For the final union bound, note that the number of pairs x, y of distance at most 3 is upper-bounded by

$$2^n \sum_{k=0}^3 \binom{n}{k} \leq 2^{n+o(n)}.$$

Thus

$$\Pr[\exists x, y : d(x, y) \leq 3, K(x, y) < \alpha_2 n] \leq 2^{n+o(n)} 2^{-c_2 n} \leq 2^{-\beta_2 n}$$

for an appropriate choice of β_2 and n large enough.

On E we have $\tilde{A} \subseteq A$, so all paths counted by the $K(x, y)$'s are entirely within A . Combining these union bounds with $\Pr[E] \leq e^{-\Omega(N)}$ and taking $\alpha = \min\{\alpha_1, \alpha_2\}$, $\beta = \min\{\beta_1, \beta_2\}$ finishes the proposition. $\square$

With our extensions of [\[HPS25\]](#) established, we are almost prepared to study Ext_A . We just need a small combinatorial lemma about the number of canonical paths using a fixed edge:

Lemma 6. *For any fixed edge e in the hypercube graph and any distance $\ell \geq 1$,*

$$|\{(u, v, i) : d(u, v) = \ell, e \in P_i(u, v)\}| \leq 6(2n)^\ell.$$

Proof. Let $D = \{j : u_j \neq v_j\}$. For e to lie on the path between u and v , it must be along a coordinate direction $i_e \in \{i\} \cup D$. So the already the number of admissible pairs (i, D) for e is at most $3n^\ell$:

- If $i \notin D$, either $i_e = i$ and there are $\binom{n-1}{\ell} \leq n^\ell$ consistent pairs (i, D) , or $i_e \in D$, in which case there are $n \binom{n-2}{\ell-1} \leq n^\ell$ consistent pairs.
- If $i \in D$, then there are $\binom{n-1}{\ell-1} \ell \leq n^\ell$ possibilities.

Now observe that the (i, D) 's can arise only from a very limited number of paths containing e ; in particular because $i_e \in \{i\} \cup D$ and the walk never alters bits in $F := (\{i\} \cup D)^c$, e fixes the bits in F for both u and v . There are thus at most 2^{r+1} starting vertices u compatible with (e, i, D) , and the information (e, i, D, u) also fixes the ending vertex v . So the final count is $6(2n)^\ell$. $\square$

Proposition 15 (Extension under local escape). *Let $A \subseteq \{0, 1\}^n$ and $B := A^c$ so that local escape holds. Then*

$$\mathbf{Inf}_{\text{unif}}(\text{Ext}_A h) \leq C_\alpha \mathbf{Inf}_{\text{unif}}^{\text{int}(A)}(h) \quad \forall h : A \rightarrow \mathbb{C}.$$

Moreover, we may take $C_\alpha \leq 1100/\alpha^3$ provided $\alpha < 1$.

Proof. For notational convenience define the unweighted Dirichlet energies

$$\mathfrak{E}(h) := \sum_{x \sim y} |h(x) - h(y)|^2, \quad \mathfrak{E}_A(h) := \sum_{\substack{x \sim y \\ x, y \in A}} |h(x) - h(y)|^2.$$

It suffices to prove

$$\mathfrak{E}(\text{Ext}_A h) \leq C_\alpha \mathfrak{E}_A(h),$$

since $\mathbf{Inf}_{\text{unif}} = (4N)^{-1} \mathfrak{E}$.

Let us decompose the hypercube edges in the natural way, into the AA , AB , and BB edges. Label their contributions to $\mathfrak{E}(\text{Ext}_A h)$ by I_{AA} , I_{AB} , and I_{BB} respectively.

AA edges. The contribution is exactly $\mathfrak{E}_A(h)$:

$$I_{AA} := \mathfrak{E}_A(h).$$

AB edges. Consider some $b \in B$. The total contribution of b to AB edges is

$$\sum_{a \in \mathcal{N}_A(b)} |\text{Ext}_A h(b) - h(a)|^2 = \frac{1}{|\mathcal{N}_A(b)|} \sum_{\{a, a'\} \subset \mathcal{N}_A(b)} |h(a) - h(a')|^2,$$

(one may recognize the identity for variance).

Now, every pair $u, v \in \mathcal{N}_A(b)$ has $d(u, v) = 2$. For any path $P = (x_0 = u, x_1, \dots, x_{\ell-1}, x_\ell = v)$ in the hypercube, we may of course write

$$h(u) - h(v) = \sum_{j=1}^{\ell} (h(x_{j-1}) - h(x_j)),$$

so

$$|h(u) - h(v)|^2 \leq \ell \sum_{j=1}^{\ell} |h(x_{j-1}) - h(x_j)|^2.$$

Averaging over the $\geq \alpha n$ good canonical paths $\mathcal{P}_{a, a'}^A$ of length at most 5 between u and v yields

$$|g(u) - g(v)|^2 \leq \frac{5}{\alpha n} \sum_{P \in \mathcal{P}_{a, a'}^A} \sum_{e \in P} |h(e_{\text{start}}) - h(e_{\text{end}})|^2,$$

so in total the AB contribution to $\mathfrak{E}(\text{Ext}_A h)$ is

$$I_{AB} = \sum_{b \in B} \sum_{a \in \mathcal{N}_A(b)} |\text{Ext}_A h(b) - h(a)|^2 \leq \frac{5}{\alpha^2 n^2} \sum_{b \in B} \sum_{\{a, a'\} \subset \mathcal{N}_A(b)} \sum_{P \in \mathcal{P}_{a, a'}^A} \sum_{e \in P} |h(e_{\text{start}}) - h(e_{\text{end}})|^2,$$

where we have used $|\mathcal{N}_A(b)| \geq \alpha n$.

Note that any $\{a, a'\} \in A$, $d(a, a') = 2$, has at most two $b \in B$ such that $\{a, a'\} \subseteq \mathcal{N}_A(b)$. Thus

$$I_{AB} \leq \frac{10}{\alpha^2 n^2} \sum_{\substack{a, a' \in A \\ d(a, a')=2}} \sum_{P \in \mathcal{P}_{a, a'}^A} \sum_{e \in P} |h(e_{\text{start}}) - h(e_{\text{end}})|^2.$$

Reordering the sum and overcounting (forgetting that the paths must be good) and then applying the counting lemma, we have

$$\begin{aligned}
I_{AB} &\leq \frac{10}{\alpha^2 n^2} \sum_{e \subset A} |\{(a, a', i) : d(a, a') = 2, e \in P_i(a, a')\}| \cdot |h(e_{\text{start}}) - h(e_{\text{end}})|^2 \\
&\leq \frac{120n^2}{\alpha^2 n^2} \sum_{x \sim y \in A} |h(x) - h(y)|^2 \\
&= \frac{120}{\alpha^2} \mathfrak{E}_A(h).
\end{aligned} \tag{Lemma 6}$$

BB edges. If $b \sim b'$

$$\begin{aligned}
|\text{Ext}_A h(b) - \text{Ext}_A h(b')|^2 &= \left| \frac{1}{|\mathcal{N}_A(b)| \cdot |\mathcal{N}_A(b')|} \sum_{u \in \mathcal{N}_A(b)} \sum_{v \in \mathcal{N}_A(b')} (h(u) - h(v)) \right|^2 \\
&\leq \frac{1}{\alpha^2 n^2} \sum_{u \in \mathcal{N}_A(b)} \sum_{v \in \mathcal{N}_A(b')} |h(u) - h(v)|^2.
\end{aligned}$$

Now $d(u, v) \in \{1, 3\}$.

For the $d(u, v) = 1$ pairs, we get a contribution of at most

$$\frac{1}{\alpha^2 n^2} \sum_{b \sim b' \in B} \sum_{\substack{u \in \mathcal{N}_A(b) \\ v \in \mathcal{N}_A(b') \\ u \sim v}} |h(u) - h(v)|^2 = \frac{1}{\alpha^2 n^2} \sum_{\substack{u \in \mathcal{N}_A(b) \\ v \in \mathcal{N}_A(b') \\ u \sim v}} |h(u) - h(v)|^2 \leq \frac{1}{\alpha^2 n} \mathfrak{E}_A(h).$$

For the $d(u, v) = 3$ part, we follow the idea from the I_{AB} edges: average over the $\geq \alpha n$ canonical good paths of length at most 5 connecting u and v :

$$|h(u) - h(v)|^2 \leq \frac{5}{\alpha n} \sum_{P \in \mathcal{P}_{u,v}^A} \sum_{e \in P} |h(e_{\text{start}}) - h(e_{\text{end}})|^2.$$

Note that any $\{a, a'\} \in A, d(a, a') = 3$, has at most four BB edges $b \sim b' \in B$ with $a \in \mathcal{N}_A(b)$ and $a' \in \mathcal{N}_A(b')$. Using this observation and applying [Lemma 6](#), we get a total contribution to I_{BB} by the $d(u, b)$ part of at most

$$\frac{20}{\alpha^3 n^3} \cdot 48n^3 \sum_{x \sim y \in A} |h(x) - h(y)|^2 = \frac{960}{\alpha^3} \mathfrak{E}_A.$$

Combining the AA , AB , and BB contributions gives

$$\mathfrak{E}(\text{Ext}_A h) \leq \left(\frac{960}{\alpha^3} + \frac{120}{\alpha^2} + 1 + o(1) \right) \mathfrak{E}_A(h). \quad \square$$

We are finally able to give the metric entropy bound.

Proof of [Theorem 8](#). Consider q_A for $q \in Q_\kappa$. By [Proposition 15](#),

$$\mathcal{E}_{\text{unif}}(\text{Ext}_A q_A) \leq C_\alpha \mathcal{E}_{\text{unif}}^{\text{int}(A)}(q_A).$$

Also, on every edge inside A ,

$$\frac{\mu(x)\mu(y)}{\mu(x) + \mu(y)} \geq \frac{1}{2TN},$$

hence

$$\mathcal{E}_{\text{unif}}^{\text{int}(A)}(q_A) \leq T\mathcal{E}_{\mu}^{\text{int}(A)}(q) \leq T\kappa.$$

Therefore

$$\mathcal{E}_{\text{unif}}(\text{Ext}_A q_A) \leq C_{\alpha} T\kappa.$$

Let $(\text{Ext}_A q_A)_{\leq d}$ be the projection of $\text{Ext}_A q_A$ onto Fourier degree at most d . Since the normalized cube Laplacian has eigenvalue $|S|/n$ on Fourier level S ,

$$\|\text{Ext}_A q_A - (\text{Ext}_A q_A)_{\leq d}\|_{L^2(\text{unif})}^2 \leq \frac{n}{d} \mathcal{E}_{\text{unif}}(\text{Ext}_A q_A) \leq C_{\alpha} T \frac{n}{d} \kappa.$$

Restricting back to A and using $\mu(x) \leq T/N$ on A ,

$$\|q_A - ((\text{Ext}_A q_A)_{\leq d})_A\|_{L^2(\mu)}^2 \leq T \|\text{Ext}_A q_A - (\text{Ext}_A q_A)_{\leq d}\|_{L^2(\text{unif})}^2 \leq C_{\alpha} T^2 \frac{n}{d} \kappa.$$

Choose

$$d = C_{\alpha} T^2 n \frac{\kappa}{\varepsilon^2}.$$

Then every q_A for $q \in Q_{\kappa}$ is within ε in $L^2(\mu)$ of the restriction to A of a degree- $\leq d$ Fourier polynomial.

The dimension of the degree- $\leq d$ Fourier space is

$$D = \sum_{j \leq d} \binom{n}{j} \leq N^{H(d/n)+o(1)},$$

where $H(\cdot)$ is binary entropy. A standard volumetric bound gives an ε -net of cardinality at most $(C/\varepsilon)^D$, hence

$$\log \mathcal{N}_{\varepsilon}(Q_{\kappa}, L^2(\mu)) \leq d \log(C/\varepsilon) \leq N^{H(\gamma)+o(1)} \log(C/\varepsilon),$$

with $\gamma = d/n = C_{\alpha} T^2 \kappa / \varepsilon^2 \leq 1100 T^2 \kappa / (\varepsilon^2 \alpha^3)$. $\square$

A corollary to this general metric entropy argument is a slightly more technical statement, that guarantees a net comprising polynomials with a certain flatness guarantee that will be needed in the sequel.

Corollary 6 (Compatible flat net for the centered A -supported class). *Let*

$$\mathcal{Q}_{\kappa, A} := \left\{ q : \text{supp}(q) \subseteq A, \mathbf{E}_{\mu} q = 0, \|q\|_{L^2(\mu)} = 1, \mathbf{Inf}_{\mu}^{\text{int}(A)}(q) \leq \kappa n \right\}.$$

Suppose that

$$H(\gamma) < \frac{1}{3}, \quad \gamma := C \frac{T^5 \kappa}{\varepsilon^2}.$$

Then there exists a finite family

$$\mathcal{Q}_{\kappa, A}^{(\varepsilon)}$$

such that every $q \in \mathcal{Q}_{\kappa, A}$ has some $v \in \mathcal{Q}_{\kappa, A}^{(\varepsilon)}$ satisfying

$$\|q - v\|_{L^2(\mu)} \leq \varepsilon.$$

Every $v \in \mathcal{Q}_{\kappa, A}^{(\varepsilon)}$ is μ -compatible and satisfies

$$\max_x \mu(x) |v(x)|^2 \leq \|\mu\|_{\infty}^{1/3}.$$

Moreover,

$$\log \left| \mathcal{Q}_{\kappa, A}^{(\varepsilon)} \right| \leq N^{H(\gamma)+o(1)} \log \left(\frac{C}{\varepsilon} \right).$$

7 Concentration of $\|\widehat{M}_{\widehat{f}}\|_{\text{op}}$ for the small subspace B

Now that we have been able to analyze the contribution from the “good” A -part, we will focus on the “bad” B -part. It will be convenient to work in the g -picture rather than the q -picture. Recall that $\widehat{M}_{\widehat{f}}$ is defined via

$$\langle g, \widehat{M}_{\widehat{f}} g \rangle = \mathbf{Avg}_{\widehat{f}}[\widehat{g}].$$

This section is devoted to a proof of the following concentration bounds.

Theorem 9. *For any $\varepsilon > 0$, there is a $T \geq 1$ such that with probability $1 - \exp(-\Omega_\varepsilon(n))$ over Haar-random f ,*

$$\|\Pi_B \widehat{M}_{\widehat{f}} \Pi_B\|_{\text{op}} \leq \frac{1}{2} + \varepsilon \quad (25)$$

$$\text{and} \quad \|\Pi_A \widehat{M}_{\widehat{f}} \Pi_B\|_{\text{op}} = \|\Pi_B \widehat{M}_{\widehat{f}} \Pi_A\|_{\text{op}} \leq \frac{1}{4} + \varepsilon, \quad (26)$$

where Π_B is the coordinate projection onto $B = B(T) = \{x : \mu_f(x) \notin [1/TN, T/N]\}$ and Π_A is onto $A = B^c$.

The proof of [Theorem 9](#) proceeds roughly as follows. Let us start with the bound for $\Pi_B \widehat{M}_{\widehat{f}} \Pi_B$ and begin by fixing a $\mu \sim \boldsymbol{\mu}$. With high probability, μ is well-behaved in a certain sense required by comparisons below. Now, notice that because we only need the good event in [Theorem 9](#) to occur with probability $1 - \exp(-\Omega(n))$, it suffices to consider tracial moments of $\Pi_B \widehat{M}_{\widehat{f}} \Pi_B$ only up to order $\mathcal{O}(n)$, rather than $N = 2^n$ (here the tracial moments are integrated over $f \sim F_\mu$ for the fixed μ). Next, we observe that from the perspective of moments of this relatively low order, when μ is “nice,” the Fourier coefficients $\widehat{f}(s)$ are close to i.i.d., and in particular they are well approximated by a product of i.i.d. Gaussians which are independent of μ entirely. In this i.i.d. Gaussian model we can then uncondition on μ , which leads to B varying as a random subset independent of the Gaussianized $\widehat{f}(s)$ ’s. This means that in $\widehat{M}_{\widehat{f}}$, the projector Π_B becomes a *uniformly random coordinate projection* of a certain size. Thus, fixing our Gaussianized $\widehat{M}_{\widehat{f}}$ and allowing B to vary, $\Pi_B \widehat{M}_{\widehat{f}} \Pi_B$ is (after centering) precisely a *random paving*, for which we appeal to the work of Tropp [\[Tro08\]](#). Provided certain technical conditions hold, [\[Tro08\]](#) states that sufficiently sparse random coordinate projections of an entrywise small matrix have small operator norm.

The proof of the BB bound, [Equation \(25\)](#), is the subject of [Section 7.2](#) and in particular [Proposition 17](#). The argument for the bound on $\|\Pi_A \widehat{M}_{\widehat{f}} \Pi_B\|_{\text{op}}$, given in [Section 7.3](#), is nearly identical except that we pass to the square of $\Pi_A \widehat{M}_{\widehat{f}} \Pi_B$, controlled by (a centered version of) $\Pi_B \widehat{M}_{\widehat{f}}^2 \Pi_B$. This object is then treated with the same argument as above.

Several technical steps are required to execute these proofs, but we can borrow much of the machinery developed in [Section 5.3](#), especially the regularization approach and corresponding derivative bounds on terms in $\widehat{M}_{\widehat{f}}$.

7.1 Technical preparations

7.1.1 Regularity of B

Proposition 16. *Set*

$$\beta_T := \Pr[\text{Exp}(1) \notin [1/T, T]] = 1 - e^{-1/T} + e^{-T} = \mathbf{E}|B|/N.$$

Then, with probability at least $1 - 2^{-\Omega(n)}$,

$$\frac{|B|}{N} = \beta_T + o(1),$$

Moreover, the law of B is exchangeable under permutations of the cube. Consequently, conditional on $|B| = k$, the set B is uniformly distributed among all k -subsets of $\{0, 1\}^n$. In particular, for every $\beta > 0$, if T is chosen sufficiently large, then with probability at least $1 - 2^{-\Omega(n)}$,

$$|B| \leq \beta N.$$

7.1.2 Random paving

We will need the following random paving estimate of Tropp [Tro08]. Let us call by a *uniform coordinate projection of density η* the random diagonal matrix R where the diagonal is distributed uniformly over 0-1 vectors with 1-density η . For any matrix A , the projection RAR is a *random paving*.

Theorem 10 (Moments of random pavings [Tro08, Prop. 4 & Thm. 5]). *Fix $\alpha > 0$ and $\varepsilon \in (0, 1)$. There exists $\beta_{\max} = \beta_{\max}(\varepsilon, \alpha) > 0$ such that the following holds for any sufficiently-large integer N .*

If A is a deterministic $N \times N$ matrix satisfying

$$\max_{x,y} |A_{xy}| \leq \frac{1}{(\log N)^{1+\alpha}} \|A\|_{\text{op}},$$

and R is a uniform coordinate projection of density at most $\beta_{\max}$, then for all even integers p with $2 \leq p \leq 2\lceil \log N \rceil$,

$$(\mathbf{E}_R \|RAR\|_{\text{op}}^p)^{1/p} \leq \varepsilon \|A\|_{\text{op}}.$$

7.2 Proof of Equation (25), the BB bound

Step 1: Regularize. Recall that

$$\widehat{M}_{\widehat{f}} = H^{\otimes n} \cdot \frac{1}{2n} \sum_{i,s} \frac{|u_{s,i}\rangle\langle u_{s,i}|}{\|u_{s,i}\|_2^2} \cdot H^{\otimes n},$$

where $|u_{s,i}\rangle = \widehat{f}(s)|s\rangle + \widehat{f}(s+i)|s+i\rangle$. Similarly to what was done in Section 5, define for $\eta > 0$ the regularized operator

$$\widehat{M}_{\widehat{f}}^{(\eta)} = H^{\otimes n} \cdot \frac{1}{2n} \sum_{i,s} \frac{|u_{s,i}\rangle\langle u_{s,i}|}{\|u_{s,i}\|_2^2 + \eta/N} \cdot H^{\otimes n}.$$

One may calculate that

$$0 \preceq \widehat{M}_{\widehat{f}} - \widehat{M}_{\widehat{f}}^{(\eta)} \preceq H^{\otimes n} \cdot \frac{1}{2n} \sum_{i,s} \frac{(\eta/N)I_{s,s+i}}{\|u_{s,i}\|_2^2 + \eta/N} \cdot H^{\otimes n} =: D_{\widehat{f}}^{(\eta)}$$

where $I_{s,s+i}$ denotes the all-zeros matrix except with I on the $(s, s+i)$ submatrix.

Thus

$$\|\Pi_B \widehat{M}_{\widehat{f}} \Pi_B\|_{\text{op}} \leq \|\Pi_B \widehat{M}_{\widehat{f}}^{(\eta)} \Pi_B\|_{\text{op}} + \|\Pi_B D_{\widehat{f}}^{(\eta)} \Pi_B\|_{\text{op}}$$

and (25) follows if we show:

Proposition 17. *For any $\varepsilon > 0$, there is a threshold $T > 0$ and smoothing parameter $\eta > 0$ such that with $B = B(T)$ defined as in Theorem 9, with probability $1 - \exp(-\Omega(n))$ over Haar-random f ,*

$$\|\Pi_B \widehat{M}_{\widehat{f}}^{(\eta)} \Pi_B\|_{\text{op}} \leq \frac{1}{2} + \varepsilon \tag{27}$$

$$\text{and} \quad \|\Pi_B D_{\widehat{f}}^{(\eta)} \Pi_B\|_{\text{op}} \leq \varepsilon. \tag{28}$$

To accomplish this, we will compare both $\widehat{M}_{\widehat{f}}^{(\eta)}$ and $D_{\widehat{f}}^{(\eta)}$ to Gaussian versions, decouple from B , and apply random paving.

Step 2: Gaussianize. With $\gamma = (\gamma_s)_s$, $s \in \{0, 1\}^n$, a family of i.i.d. standard complex Gaussians with variance $1/N$, define $\widehat{M}_\gamma^{(\eta)}$ and $D_\gamma^{(\eta)}$ in analogy to $\widehat{M}_{\widehat{f}}^{(\eta)}$ and $D_{\widehat{f}}^{(\eta)}$. That is, define $|g_{s,i}\rangle = \gamma_s |s\rangle + \gamma_{s+i} |s+i\rangle$ and

$$\widehat{M}_\gamma^{(\eta)} = H^{\otimes n} \cdot \frac{1}{2n} \sum_{i,s} \frac{|g_{s,i}\rangle \langle g_{s,i}|}{\|g_{s,i}\|_2^2 + \eta/N} \cdot H^{\otimes n} \quad \text{and} \quad D_\gamma^{(\eta)} = H^{\otimes n} \cdot \frac{1}{2n} \sum_{i,s} \frac{(\eta/N) I_{s,s+i}}{\|g_{s,i}\|_2^2 + \eta/N} \cdot H^{\otimes n}.$$

Also define the scalars

$$m_\eta = \mathbf{E} \frac{|\gamma_1|^2}{|\gamma_1|^2 + |\gamma_2|^2 + \eta/N} \quad \text{and} \quad d_\eta = \mathbf{E} \frac{\eta/N}{|\gamma_1|^2 + |\gamma_2|^2 + \eta/N} = 1 - 2m_\eta, \quad (29)$$

which will eventually be used to center the Gaussianized operators. Then we have the following.

Proposition 18 (*BB Gaussianization*). *Fix $\eta > 0$ and a measure μ . Then there exists $C_\eta > 0$ such that for any even $p \geq 8$ satisfying*

$$\text{Bias}(\mu)^{2/p} \leq \frac{1}{2},$$

the differences

$$\left| \left(\mathbf{E} \text{tr} |\Pi_B(\widehat{M}_{\widehat{f}}^{(\eta)} - m_\eta I) \Pi_B|^p \right)^{1/p} - \left(\mathbf{E} \text{tr} |\Pi_B(\widehat{M}_\gamma^{(\eta)} - m_\eta I) \Pi_B|^p \right)^{1/p} \right|$$

and

$$\left| \left(\mathbf{E} \text{tr} |\Pi_B(D_{\widehat{f}}^{(\eta)} - d_\eta I) \Pi_B|^p \right)^{1/p} - \left(\mathbf{E} \text{tr} |\Pi_B(D_\gamma^{(\eta)} - d_\eta I) \Pi_B|^p \right)^{1/p} \right|$$

are at most

$$C_\eta N^{1/p} \left[p^3 \|\mu\|_{3/2}^{1/2} + \left(\frac{|B|}{N} + \max_{y \in \{0,1\}^n} \mu(B+y) \right)^{1/2} + \text{Bias}(\mu)^{3/p} \right].$$

Here $B+y$ denotes the set $\{b+y : b \in B\}$.

Proof. Throughout the proof, C_η denotes a constant depending only on η , whose value may change from line to line.

Let us begin with the $\widehat{M}_{\widehat{f}}^{(\eta)}$ estimate. We will proceed in two steps: first we do a Gaussian replacement of the Steinhaus-random vector f via a Lindeberg-type argument. Then, we move to Fourier space and consider the quantity of interest as a function of the random vector $(\widehat{f}(s))_s$. With this perspective, we argue that small tracial moments of the corresponding quantity behave the same as if the Fourier coefficients $(\widehat{f}(s))_s$ were actually i.i.d. Gaussians.

Steinhaus to physical Gaussians. Let $\xi = (\xi_x)_x$, where ξ_x , $x \in \{0, 1\}^n$ are independent complex Gaussians with variances $\mu(x)$ respectively. We compare $\widehat{M}_{\widehat{f}}^{(\eta)}$ and $\widehat{M}_\xi^{(\eta)}$ by replacing the physical coordinates one at a time.

Choose a physical coordinate x , fix a $\tilde{z} \in \mathbb{C}^{2^n}$, and let $z = z(\zeta)$ be obtained from $\tilde{z}$ by replacing its x -th coordinate with $\sqrt{\mu(x)} \zeta$. Define

$$W_x(\zeta) := \Pi_B \left(\widehat{M}_{z(\zeta)}^{(\eta)} - m_\eta I \right) \Pi_B.$$

Note that $0 \preceq \widehat{M}_{z(\zeta)}^{(\eta)} \preceq I$, so

$$\|W_x(\zeta)\|_{\text{op}} \leq 1.$$

Since

$$D_\zeta \widehat{z(\zeta)}(s)[v] = \sqrt{\frac{\mu(x)}{N}} (-1)^{x \cdot s} v,$$

[Lemma 5](#) applied with regularization parameter η/N , along with the chain rule, give for $r = 1, 2, 3$,

$$\begin{aligned} \|D_\zeta^r W_x(\zeta)\| &\leq \frac{1}{n} \sum_i \max_{s: s_i=0} \left\| D_\zeta^r \frac{|u_{s,i}(\zeta)\rangle\langle u_{s,i}(\zeta)|}{\|u_{s,i}(\zeta)\|_2^2 + \eta/N} \right\| \\ &\lesssim_r \left(\frac{N}{\eta}\right)^{r/2} \left(\frac{\mu(x)}{N}\right)^{r/2} \\ &= \eta^{-r/2} \mu(x)^{r/2}, \end{aligned}$$

where $|u_{s,i}(\zeta)\rangle := \widehat{z(\zeta)}(s) |s\rangle + \widehat{z(\zeta)}(s+i) |s+i\rangle$.

Let p be even. Differentiating $\text{tr}[W_x(\zeta)^p]$ three times and applying Schatten Hölder to the three possible types of product-rule terms gives

$$\begin{aligned} \|D_\zeta^3 \text{tr}[W_x(\zeta)^p]\| &\leq C_\eta \mu(x)^{3/2} \left[p N^{1/p} \|W_x(\zeta)\|_{S_p}^{p-1} \right. \\ &\quad \left. + p^2 N^{2/p} \|W_x(\zeta)\|_{S_p}^{p-2} \right. \\ &\quad \left. + p^3 N^{3/p} \|W_x(\zeta)\|_{S_p}^{p-3} \right], \end{aligned} \tag{30}$$

where $\|\cdot\|_p$ denotes the p th Schatten norm.

We now perform the Lindeberg replacement at the level of the p -th moment roots. Enumerate the physical coordinates as $x_1, \dots, x_N$, and let $f^{(k)}$ be the hybrid vector in which the first k coordinates use Steinhaus variables and the remaining coordinates use Gaussian variables; so for example $f^{(0)} = \xi$ and $f^{(N)} = f$.

Choose K_η sufficiently large and set

$$\Lambda := K_\eta p^3 \|\mu\|_{3/2}^{1/2}.$$

For $0 \leq k \leq N$, put

$$R_k := \left(N \Lambda^p + \mathbf{E} \text{tr} \left| \Pi_B \left(\widehat{M}_{f^{(k)}}^{(\eta)} - m_\eta I \right) \Pi_B \right|^p \right)^{1/p},$$

and let

$$R_* := \max_{0 \leq k \leq N} R_k.$$

In particular,

$$R_* \geq N^{1/p} \Lambda. \tag{31}$$

Fix one replacement coordinate x . The first derivative bound above gives

$$\|W_x(t\zeta) - W_x(0)\|_{S_p} \leq C_\eta N^{1/p} \sqrt{\mu(x)} |\zeta|. \tag{32}$$

Moreover, Minkowski's inequality and the Gaussian moment bound $(\mathbf{E} |g|^p)^{1/p} \lesssim \sqrt{p}$ give

$$\begin{aligned} &\left(N \Lambda^p + \mathbf{E} \|W_{x,0}\|_{S_p}^p \right)^{1/p} \\ &\leq R_* + C_\eta N^{1/p} \sqrt{p \mu(x)} \\ &\leq R_* \left(1 + \frac{C_\eta}{K_\eta p^{5/2}} \right), \end{aligned} \tag{33}$$

where we also used $\sqrt{\mu(x)} \leq \|\mu\|_{3/2}^{1/2}$.

The pointwise quantity

$$\left(N\Lambda^p + \|W_{x,0}\|_{S_p}^p\right)^{1/p}$$

is at least $N^{1/p}\Lambda$. Combining this observation with (32) gives, for $j = 1, 2, 3$,

$$\|W_x(t\zeta)\|_{S_p}^{p-j} \leq \left(N\Lambda^p + \|W_{x,0}\|_{S_p}^p\right)^{(p-j)/p} \exp\left(\frac{C_\eta p \sqrt{\mu(x)}}{\Lambda} |\zeta|\right).$$

Since

$$\frac{p\sqrt{\mu(x)}}{\Lambda} \leq \frac{1}{K_\eta p^2},$$

(33) and the exponential moments of a standard complex Gaussian imply

$$\mathbf{E} \left[|\zeta|^3 \sup_{0 \leq t \leq 1} \|W_x(t\zeta)\|_{S_p}^{p-j} \right] \leq C_\eta R_*^{p-j}, \quad j = 1, 2, 3, \quad (34)$$

for both a Steinhaus variable and a standard complex Gaussian.

Taylor expansion of $|W_x(\cdot)|^p$ to second order at zero, together with the matching first two real moments of the Steinhaus and complex Gaussian variables, now gives

$$\begin{aligned} & \left| \mathbf{E} \operatorname{tr} \left| \Pi_B \left(\widehat{M}_{\frac{f^{(k)}}{f^{(k)}}}^{(\eta)} - m_\eta I \right) \Pi_B \right|^p - \mathbf{E} \operatorname{tr} \left| \Pi_B \left(\widehat{M}_{\frac{f^{(k-1)}}{f^{(k-1)}}}^{(\eta)} - m_\eta I \right) \Pi_B \right|^p \right| \\ & \leq C_\eta \mu(x_k)^{3/2} \left[p N^{1/p} R_*^{p-1} + p^2 N^{2/p} R_*^{p-2} + p^3 N^{3/p} R_*^{p-3} \right]. \end{aligned} \quad (35)$$

Let k_* be such that $R_{k_*} = R_*$. Telescoping from either endpoint to k_* and summing (35) over the physical coordinates gives, for $j \in \{0, N\}$,

$$R_* - R_j \leq C_\eta N^{1/p} \left[p \sum_x \mu(x)^{3/2} + \frac{p^2}{\Lambda} \sum_x \mu(x)^{3/2} + \frac{p^3}{\Lambda^2} \sum_x \mu(x)^{3/2} \right]. \quad (36)$$

Using the definition of Λ , the fact that $\sum_x \mu(x)^{3/2} \leq 1$, and

$$\|\mu\|_{3/2}^{1/2} = \left(\sum_x \mu(x)^{3/2} \right)^{1/3},$$

we obtain

$$R_* - R_j \leq C_\eta N^{1/p} p^3 \|\mu\|_{3/2}^{1/2}.$$

Removing the temporary floor $N\Lambda^p$ and using the last estimate for both endpoints gives the desired comparison to physical Gaussians:

$$\begin{aligned} & \left| \left(\mathbf{E} \operatorname{tr} \left| \Pi_B \left(\widehat{M}_{\frac{f}{f}}^{(\eta)} - m_\eta I \right) \Pi_B \right|^p \right)^{1/p} - \left(\mathbf{E} \operatorname{tr} \left| \Pi_B \left(\widehat{M}_{\frac{\xi}{\xi}}^{(\eta)} - m_\eta I \right) \Pi_B \right|^p \right)^{1/p} \right| \\ & \leq C_\eta N^{1/p} p^3 \|\mu\|_{3/2}^{1/2}. \end{aligned} \quad (37)$$

Physical Gaussians to independent Fourier Gaussians. Next we pass from $\widehat{M}_{\hat{\xi}}^{(\eta)}$ to $\widehat{M}_{\gamma}^{(\eta)}$ by Gaussian interpolation. Let $X(\tau)$, $0 \leq \tau \leq 1$ be a Gaussian vector interpolating between $(\hat{\xi}(s))_s$ at $X(0)$ and the $(\gamma(s))_s$ at $X(1)$. That is, with Γ_0 the covariance of $\hat{\xi}$ and $\Gamma_1 = N^{-1}I$ the covariance of γ , define $X(\tau)$ to have covariance $(1 - \tau)\Gamma_0 + \tau\Gamma_1$. Also define

$$\Delta_{st} := (\Gamma_1 - \Gamma_0)_{st}.$$

The covariances of $\hat{\xi}$ are $1/N$ on the diagonal, while for $s \neq t$,

$$|(\Gamma_0)_{st}| = \left| \mathbf{E} \widehat{\xi}(s) \overline{\widehat{\xi}(t)} \right| = \left| \frac{1}{N} \sum_x \mu(x) (-1)^{(s+t) \cdot x} \right| \leq \frac{\text{Bias}(\mu)}{N}.$$

Thus

$$\Delta_{ss} = 0 \quad \text{and} \quad |\Delta_{st}| \leq \frac{\text{Bias}(\mu)}{N} \quad \text{for all } s \neq t. \quad (38)$$

Define

$$W := W(\tau) := \Pi_B \left(\widehat{M}_{X(\tau)}^{(\eta)} - m_\eta I \right) \Pi_B.$$

Since $W(\tau)$ is self-adjoint and p is even,

$$\text{tr} |W(\tau)|^p = \text{tr} [W(\tau)^p].$$

The fundamental theorem of calculus and Gaussian covariance interpolation give

$$\begin{aligned} & \mathbf{E} \text{tr} \left| \Pi_B \left(\widehat{M}_{\gamma}^{(\eta)} - m_\eta I \right) \Pi_B \right|^p - \mathbf{E} \text{tr} \left| \Pi_B \left(\widehat{M}_{\hat{\xi}}^{(\eta)} - m_\eta I \right) \Pi_B \right|^p \\ &= \int_0^1 \frac{d}{d\tau} \mathbf{E} \text{tr} [W(\tau)^p] d\tau \\ &= \int_0^1 \mathbf{E} \sum_{s,t} \Delta_{st} \partial_s \bar{\partial}_t \text{tr} [W(\tau)^p] d\tau \\ &= \int_0^1 \mathbf{E} \sum_{s \neq t} \Delta_{st} \partial_s \bar{\partial}_t \text{tr} [W(\tau)^p] d\tau. \end{aligned}$$

Here ∂_s and $\bar{\partial}_t$ denote the Wirtinger derivatives in the Fourier coordinates:

$$\partial_s := \frac{\partial}{\partial z_s}, \quad \bar{\partial}_t := \frac{\partial}{\partial \bar{z}_t}.$$

We now pursue a uniform bound on the interpolation integrand. By the product rule and cyclicity of trace,

$$\partial_s \bar{\partial}_t \text{tr} [W^p] = \underbrace{p \text{tr} [(\partial_s \bar{\partial}_t W) W^{p-1}]}_{=: T_1(s,t)} + p \sum_{\substack{\ell, \ell' \geq 0 \\ \ell + \ell' = p-2}} \underbrace{\text{tr} [(\partial_s W) W^\ell (\bar{\partial}_t W) W^{\ell'}]}_{=: T_2(s,t,\ell)}. \quad (39)$$

We first control the covariance-weighted T_1 terms. We then treat separately the interior T_2 terms, for which $1 \leq \ell \leq p-3$, and the two endpoint terms $\ell = 0, p-2$.

The T_1 terms. Recall that

$$W(\tau) = \Pi_B H^{\otimes n} \left(\frac{1}{2n} \sum_{i,s} \frac{|u_{s,i}\rangle\langle u_{s,i}|}{\|u_{s,i}\|_2^2 + \eta/N} - m_\eta I \right) H^{\otimes n} \Pi_B,$$

where $|u_{s,i}\rangle = X(\tau)_s |s\rangle + X(\tau)_{s+e_i} |s+e_i\rangle$. Since $\Delta_{ss} = 0$, only mixed derivatives with respect to distinct Fourier coordinates contribute, and for a fixed direction i , such a mixed derivative is nonzero only when the two coordinates are the endpoints of a common i -edge. Grouping the two directed copies of each i -edge, we therefore have

$$\begin{aligned} \sum_{r,t} \Delta_{rt} \partial_r \bar{\partial}_t W &= \Pi_B H^{\otimes n} \left[\frac{1}{n} \sum_i \bigoplus_{s:s_i=0} \left(\Delta_{s,s+e_i} \partial_s \bar{\partial}_{s+e_i} \frac{|u_{s,i}\rangle\langle u_{s,i}|}{\|u_{s,i}\|_2^2 + \eta/N} \right. \right. \\ &\quad \left. \left. + \Delta_{s+e_i,s} \partial_{s+e_i} \bar{\partial}_s \frac{|u_{s,i}\rangle\langle u_{s,i}|}{\|u_{s,i}\|_2^2 + \eta/N} \right) \right] H^{\otimes n} \Pi_B. \end{aligned}$$

By [Lemma 5](#), each local mixed derivative in this display has operator norm at most $C_\eta N$. Using the direct-sum structure, unitarity of $H^{\otimes n}$, contractivity of Π_B , and $|\Delta_{rt}| \leq \text{Bias}(\mu)/N$, we obtain

$$\begin{aligned} \left\| \sum_{r,t} \Delta_{rt} \partial_r \bar{\partial}_t W \right\|_{\text{op}} &\leq \frac{1}{n} \sum_i \max_{s:s_i=0} \left[\left\| \Delta_{s,s+e_i} \partial_s \bar{\partial}_{s+e_i} \frac{|u_{s,i}\rangle\langle u_{s,i}|}{\|u_{s,i}\|_2^2 + \eta/N} \right\|_{\text{op}} \right. \\ &\quad \left. + \left\| \Delta_{s+e_i,s} \partial_{s+e_i} \bar{\partial}_s \frac{|u_{s,i}\rangle\langle u_{s,i}|}{\|u_{s,i}\|_2^2 + \eta/N} \right\|_{\text{op}} \right] \\ &\leq C_\eta \text{Bias}(\mu). \end{aligned} \tag{40}$$

Consequently, Schatten Hölder gives

$$\begin{aligned} \left| \sum_{s,t} \Delta_{st} T_1(s,t) \right| &= \left| \text{tr} \left[\left(\sum_{s,t} \Delta_{st} \partial_s \bar{\partial}_t W \right) W^{p-1} \right] \right| \\ &\leq \left\| \sum_{s,t} \Delta_{st} \partial_s \bar{\partial}_t W \right\|_{S_p} \|W^{p-1}\|_{S_{p/(p-1)}} \\ &\leq C_\eta N^{1/p} \text{Bias}(\mu) \|W\|_{S_p}^{p-1}. \end{aligned} \tag{41}$$

The T_2 terms. Fix ℓ and $\ell' = p-2-\ell$. For this fixed choice,

$$\begin{aligned} \sum_{s,t} \Delta_{st} T_2(s,t,\ell) &= \sum_{s,t} \Delta_{st} \text{tr} \left[(\partial_s W) W^\ell (\bar{\partial}_t W) W^{\ell'} \right] \\ &= \sum_{s,t} \Delta_{st} \text{tr} \left[(\partial_s M_{X(\tau)}^{(\eta)}) \underbrace{H^{\otimes n} \Pi_B W^\ell \Pi_B H^{\otimes n}}_{=: Q} (\bar{\partial}_t M_{X(\tau)}^{(\eta)}) \underbrace{H^{\otimes n} \Pi_B W^{\ell'} \Pi_B H^{\otimes n}}_{=: Q'} \right]. \end{aligned}$$

Expand the Fourier-side operator into its 2×2 edge blocks:

$$M_{X(\tau)}^{(\eta)} = \frac{1}{2n} \sum_i \sum_r \sum_{a,b \in \{0,1\}} c_{r,a,b,i} |r+ae_i\rangle\langle r+be_i|.$$

The derivative $\partial_s c_{r,a,b,i}$ vanishes unless $r=s$ or $r=s+e_i$. Define

$$\begin{aligned} c'_{s,a,b,i} &:= \partial_s c_{s,a,b,i} + \partial_s c_{s+e_i,a+1,b+1,i}, \\ \tilde{c}'_{t,c,d,j} \text{ and } &:= \bar{\partial}_t c_{t,c,d,j} + \bar{\partial}_t c_{t+e_j,c+1,d+1,j}, \end{aligned}$$

where the additions in a, b, c, d are modulo 2. With these definitions,

$$\partial_s M_{X(\tau)}^{(\eta)} = \frac{1}{2n} \sum_i \sum_{a,b \in \{0,1\}} c'_{s,a,b,i} |s + ae_i\rangle \langle s + be_i|,$$

and

$$\bar{\partial}_t M_{X(\tau)}^{(\eta)} = \frac{1}{2n} \sum_j \sum_{c,d \in \{0,1\}} \tilde{c}'_{t,c,d,j} |t + ce_j\rangle \langle t + de_j|.$$

By [Lemma 5](#),

$$|c'_{s,a,b,i}|, |\tilde{c}'_{t,c,d,j}| \leq C_\eta \sqrt{N}. \quad (42)$$

Substituting these expansions into the preceding covariance-weighted sum gives

$$\begin{aligned} \sum_{s,t} \Delta_{st} T_2(s, t, \ell) &= \frac{1}{4n^2} \sum_{i,j} \sum_{a,b,c,d \in \{0,1\}} \sum_{s,t} \Delta_{st} c'_{s,a,b,i} \tilde{c}'_{t,c,d,j} \\ &\quad \cdot \text{tr} [|s + ae_i\rangle \langle s + be_i| Q |t + ce_j\rangle \langle t + de_j| Q'] \\ &= \frac{1}{4n^2} \sum_{i,j} \sum_{a,b,c,d} \sum_{s,t} \Delta_{st} c'_{s,a,b,i} \tilde{c}'_{t,c,d,j} \\ &\quad \cdot \langle t + de_j| Q' |s + ae_i\rangle \langle s + be_i| Q |t + ce_j\rangle. \end{aligned} \quad (43)$$

Interior T_2 terms. Suppose that

$$1 \leq \ell \leq p - 3.$$

Let S_u denote translation in the Fourier coordinates:

$$S_u |s\rangle := |s + u\rangle.$$

Let

$$\mathcal{S}_\Delta(A) := \Delta \circ A$$

denote Schur multiplication by Δ . Writing $\chi_x(s) := (-1)^{x \cdot s}$, we have the exact identity

$$\mathcal{S}_\Delta(A) = \frac{1}{N} \sum_x \left(\frac{1}{N} - \mu(x) \right) \text{diag}(\chi_x) A \text{diag}(\chi_x). \quad (44)$$

Indeed, the (s, t) entry of the right-hand side is

$$\frac{1}{N} \sum_x \left(\frac{1}{N} - \mu(x) \right) (-1)^{x \cdot (s+t)} A_{st} = \Delta_{st} A_{st}.$$

Since

$$\sum_x \left| \frac{1}{N} - \mu(x) \right| \leq 2,$$

we obtain

$$\|\mathcal{S}_\Delta\|_{S_\infty \rightarrow S_\infty} \leq \frac{2}{N}.$$

Moreover, since a Schur multiplier is diagonal in the matrix-unit basis,

$$\|\mathcal{S}_\Delta\|_{S_2 \rightarrow S_2} = \max_{s,t} |\Delta_{st}| \leq \frac{\text{Bias}(\mu)}{N}.$$

Interpolation between these two bounds, together with duality, gives

$$\|\mathcal{S}_\Delta(A)\|_{S_q} \leq \frac{2}{N} \text{Bias}(\mu)^{2 \min\{\frac{1}{q}, 1-\frac{1}{q}\}} \|A\|_{S_q} \quad (45)$$

for every $1 \leq q \leq \infty$.

For fixed i, j, a, b, c, d , the inner sum over s, t in (43) is exactly

$$\text{tr} \left[\text{diag}(\tilde{c}'_{t,c,d,j})_t S_{de_j}^* Q' S_{ae_i} \cdot \mathcal{S}_\Delta(\text{diag}(c'_{s,a,b,i})_s S_{be_i}^* Q S_{ce_j}) \right]. \quad (46)$$

Indeed, expanding the trace in (46) gives

$$\sum_{s,t} \Delta_{st} c'_{s,a,b,i} \tilde{c}'_{t,c,d,j} \langle t + de_j | Q' | s + ae_i \rangle \langle s + be_i | Q | t + ce_j \rangle,$$

which is the corresponding term in (43).

We apply Schatten Hölder with the conjugate exponents

$$\frac{p}{\ell'} \quad \text{and} \quad \frac{p}{\ell + 2}.$$

The shifts are unitary, and $W = \Pi_B W \Pi_B$, so

$$\left\| \text{diag}(\tilde{c}'_{t,c,d,j})_t S_{de_j}^* Q' S_{ae_i} \right\|_{S_{p/\ell'}} \leq C_\eta \sqrt{N} \|Q'\|_{S_{p/\ell'}} = C_\eta \sqrt{N} \|W\|_{S_p}^{\ell'}. \quad (47)$$

Similarly,

$$\left\| \text{diag}(c'_{s,a,b,i})_s S_{be_i}^* Q S_{ce_j} \right\|_{S_{p/(\ell+2)}} \leq C_\eta \sqrt{N} \|W\|_{S_{p/(\ell+2)}}^\ell \leq C_\eta \sqrt{N} N^{2/p} \|W\|_{S_p}^\ell. \quad (48)$$

The last inequality is the finite-dimensional Schatten embedding

$$\|W^\ell\|_{S_{p/(\ell+2)}} \leq N^{2/p} \|W\|_{S_p}^\ell.$$

Applying Schatten Hölder to (46), and then using (45), (47), and (48), gives

$$\begin{aligned} & \left| \text{tr} \left[\text{diag}(\tilde{c}'_{t,c,d,j})_t S_{de_j}^* Q' S_{ae_i} \cdot \mathcal{S}_\Delta(\text{diag}(c'_{s,a,b,i})_s S_{be_i}^* Q S_{ce_j}) \right] \right| \\ & \leq C_\eta \text{Bias}(\mu)^{2 \min\{\frac{\ell+2}{p}, \frac{\ell'}{p}\}} N^{2/p} \|W\|_{S_p}^{p-2}. \end{aligned}$$

Here the factor $1/N$ from the Schur-multiplier estimate cancels the two $\sqrt{N}$ factors from the differentiated edge coefficients. The prefactor $1/(4n^2)$ in (43) cancels the n^2 choices of i, j , up to an absolute constant, while the choices of a, b, c, d contribute another absolute constant. Consequently,

$$\left| \sum_{s,t} \Delta_{st} T_2(s, t, \ell) \right| \leq C_\eta N^{2/p} \text{Bias}(\mu)^{2 \min\{\frac{\ell+2}{p}, \frac{\ell'}{p}\}} \|W\|_{S_p}^{p-2}. \quad (49)$$

Since Δ is symmetric,

$$\text{tr}[A \mathcal{S}_\Delta(B)] = \text{tr}[\mathcal{S}_\Delta(A)B].$$

We may therefore apply the Schur-multiplier estimate to the factor containing Q' instead. Taking the better of the two bounds gives

$$\left| \sum_{s,t} \Delta_{st} T_2(s, t, \ell) \right| \leq C_\eta N^{2/p} \text{Bias}(\mu)^{\theta_\ell} \|W\|_{S_p}^{p-2}, \quad (50)$$

where

$$\theta_\ell := \max \left\{ 2 \min \left(\frac{\ell+2}{p}, \frac{\ell'}{p} \right), 2 \min \left(\frac{\ell'+2}{p}, \frac{\ell}{p} \right) \right\}.$$

As ℓ ranges over $1, \dots, p-3$, each power $\text{Bias}(\mu)^{2k/p}$ with $k \geq 3$ occurs at most four times among the quantities $\text{Bias}(\mu)^{\theta_\ell}$. Therefore,

$$\sum_{\ell=1}^{p-3} \text{Bias}(\mu)^{\theta_\ell} \leq 4 \sum_{k=3}^{\infty} \text{Bias}(\mu)^{2k/p} = \frac{4 \text{Bias}(\mu)^{6/p}}{1 - \text{Bias}(\mu)^{2/p}} \leq 8 \text{Bias}(\mu)^{6/p}. \quad (51)$$

Endpoint T_2 terms. It remains to treat $\ell = 0$ and $\ell = p-2$. When $\ell = 0$,

$$Q = H^{\otimes n} \Pi_B H^{\otimes n}, \quad \text{and} \quad Q' = H^{\otimes n} \Pi_B W^{p-2} \Pi_B H^{\otimes n}.$$

For arbitrary shifts $u, v \in \{0, 1\}^n$, one has

$$\|S_\Delta(S_u^* H^{\otimes n} \Pi_B H^{\otimes n} S_v)\|_{\text{op}} \leq \frac{1}{N} \left(\frac{|B|}{N} + \max_y \mu(B+y) \right). \quad (52)$$

Indeed,

$$(S_u^* H^{\otimes n} \Pi_B H^{\otimes n} S_v)_{s,t} = \frac{1}{N} \sum_{y \in B} (-1)^{y \cdot (u+v)} (-1)^{y \cdot (s+t)},$$

while

$$\Delta_{s,t} = \frac{1}{N} \sum_x \left(\frac{1}{N} - \mu(x) \right) (-1)^{x \cdot (s+t)}.$$

Their Schur product is Walsh-diagonalizable, with eigenvalues

$$\frac{1}{N} \sum_x \left(\frac{1}{N} - \mu(x) \right) \mathbf{1}_B(z+x) (-1)^{(z+x) \cdot (u+v)}.$$

The absolute value of every such eigenvalue is at most

$$\frac{1}{N} \left(\frac{|B|}{N} + \mu(B+z) \right),$$

which proves (52).

Schur multiplication commutes with multiplication by a diagonal matrix on the left, so

$$\|S_\Delta(\text{diag}(c'_{s,a,b,i})_s S_{be_i}^* Q S_{ce_j})\|_{\text{op}} \leq C_\eta \sqrt{N} \frac{1}{N} \left(\frac{|B|}{N} + \max_y \mu(B+y) \right).$$

The other factor in (46) satisfies

$$\left\| \text{diag}(\tilde{c}'_{t,c,d,j})_t S_{de_j}^* Q' S_{ae_i} \right\|_{S_1} \leq C_\eta \sqrt{N} \|W^{p-2}\|_{S_1}.$$

The two $\sqrt{N}$ factors again cancel the $1/N$ in (52). Moreover,

$$\|W^{p-2}\|_{S_1} \leq N^{2/p} \|W\|_{S_p}^{p-2}.$$

Applying Schatten Hölder in (46), and then summing over i, j, a, b, c, d , gives

$$\left| \sum_{s,t} \Delta_{st} T_2(s, t, 0) \right| \leq C_\eta N^{2/p} \left(\frac{|B|}{N} + \max_y \mu(B+y) \right) \|W\|_{S_p}^{p-2}. \quad (53)$$

The endpoint $\ell = p - 2$ is identical, with the two differentiated factors interchanged.

We now combine the T_1 terms, the interior T_2 terms, and the endpoint T_2 terms. Define

$$R(\tau) := (\mathbf{E} \operatorname{tr} |W(\tau)|^p)^{1/p}.$$

For $j = 1, 2$, Hölder's inequality gives

$$\mathbf{E} \|W(\tau)\|_{S_p}^{p-j} \leq R(\tau)^{p-j}.$$

Using (41), (51), and (53) in the Gaussian interpolation identity gives

$$\begin{aligned} \left| \frac{d}{d\tau} \mathbf{E} \operatorname{tr} |W(\tau)|^p \right| &\leq C_\eta p \left[N^{1/p} \operatorname{Bias}(\mu) R(\tau)^{p-1} \right. \\ &\quad \left. + N^{2/p} \left(\frac{|B|}{N} + \max_y \mu(B+y) + \operatorname{Bias}(\mu)^{6/p} \right) R(\tau)^{p-2} \right]. \end{aligned} \quad (54)$$

We now convert this differential inequality for the p -th moment into one for its p -th root. At every point where $R(\tau) > 0$, (54) gives

$$|R'(\tau)| \leq C_\eta \left[N^{1/p} \operatorname{Bias}(\mu) + \frac{N^{2/p} \left(\frac{|B|}{N} + \max_y \mu(B+y) + \operatorname{Bias}(\mu)^{6/p} \right)}{R(\tau)} \right].$$

Consequently,

$$\begin{aligned} &\left| \frac{d}{d\tau} \left(R(\tau)^2 + N^{2/p} \left[\frac{|B|}{N} + \max_y \mu(B+y) + \operatorname{Bias}(\mu)^{6/p} \right] \right)^{1/2} \right| \\ &\leq C_\eta N^{1/p} \left[\operatorname{Bias}(\mu) + \left(\frac{|B|}{N} + \max_y \mu(B+y) + \operatorname{Bias}(\mu)^{6/p} \right)^{1/2} \right]. \end{aligned}$$

(At points where $R(\tau) = 0$, one may apply the same argument after replacing $\mathbf{E} \operatorname{tr} |W(\tau)|^p$ by $\mathbf{E} \operatorname{tr} |W(\tau)|^p + \delta^p$, and then let $\delta \downarrow 0$.) Integrating over $\tau \in [0, 1]$ gives the same bound for the difference of the square-root expressions at $\tau = 0$ and $\tau = 1$. For every τ ,

$$\begin{aligned} 0 &\leq \left(R(\tau)^2 + N^{2/p} \left[\frac{|B|}{N} + \max_y \mu(B+y) + \operatorname{Bias}(\mu)^{6/p} \right] \right)^{1/2} - R(\tau) \\ &\leq N^{1/p} \left(\frac{|B|}{N} + \max_y \mu(B+y) + \operatorname{Bias}(\mu)^{6/p} \right)^{1/2}. \end{aligned}$$

It follows that

$$|R(1) - R(0)| \leq C_\eta N^{1/p} \left[\operatorname{Bias}(\mu) + \left(\frac{|B|}{N} + \max_y \mu(B+y) + \operatorname{Bias}(\mu)^{6/p} \right)^{1/2} \right].$$

Since

$$\operatorname{Bias}(\mu) \leq \operatorname{Bias}(\mu)^{3/p}$$

and

$$\left(\frac{|B|}{N} + \max_y \mu(B+y) + \operatorname{Bias}(\mu)^{6/p} \right)^{1/2} \leq \left(\frac{|B|}{N} + \max_y \mu(B+y) \right)^{1/2} + \operatorname{Bias}(\mu)^{3/p},$$

we conclude that

$$\begin{aligned} & \left| \left(\mathbf{E} \operatorname{tr} \left| \Pi_B \left(\widehat{M}_{\xi}^{(\eta)} - m_{\eta} I \right) \Pi_B \right|^p \right)^{1/p} - \left(\mathbf{E} \operatorname{tr} \left| \Pi_B \left(\widehat{M}_{\gamma}^{(\eta)} - m_{\eta} I \right) \Pi_B \right|^p \right)^{1/p} \right| \\ & \leq C_{\eta} N^{1/p} \left[\left(\frac{|B|}{N} + \max_y \mu(B+y) \right)^{1/2} + \operatorname{Bias}(\mu)^{3/p} \right]. \end{aligned} \quad (55)$$

Combining (37) and (55) proves the asserted estimate for $\widehat{M}^{(\eta)}$.

Finally, consider $D^{(\eta)}$. On each Fourier edge, its local block is

$$Q_{\eta}(u) = \frac{\eta/N}{\|u\|_2^2 + \eta/N} I_2 = \left(1 - \operatorname{tr} \frac{uu^*}{\|u\|_2^2 + \eta/N} \right) I_2.$$

Its first three derivatives have the same scales as those of the regularized projector block:

$$\|D^r Q_{\eta}(u)\| \lesssim_r \left(\frac{N}{\eta} \right)^{r/2}, \quad r = 1, 2, 3.$$

Consequently, the root-level Lindeberg argument, the covariance-weighted T_1 estimate, the interior T_2 argument, and the endpoint T_2 argument all apply identically to

$$\Pi_B \left(D^{(\eta)} - d_{\eta} I \right) \Pi_B.$$

This proves the asserted estimate for $D^{(\eta)}$ and completes the proof. $\square$

Step 3: Random pavings. We now apply Theorem 10 to $\widehat{M}_{\gamma}^{(\eta)}$ to complete the argument for the BB case.

Proposition 19 (Gaussian random-paving bounds for the BB objects). *For every $\varepsilon, \eta > 0$, there exists a density threshold $\beta_{\max} > 0$ such that the following holds for n large enough.*

Let $\gamma = (\gamma_s)_{s \in \{0,1\}^n}$ be i.i.d. complex Gaussians with variance $1/N$ and let $B \subseteq \{0,1\}^n$ be an exchangeable random subset, independent of γ . Then for every even $p \leq n$,

$$\mathbf{E} \left[\mathbf{1}_{|B|/N \leq \beta_{\max}} \cdot \operatorname{tr} \left| \Pi_B \left(\widehat{M}_{\gamma}^{(\eta)} - m_{\eta} I \right) \Pi_B \right|^p \right]^{1/p} \leq N^{1/p} \varepsilon$$

and

$$\mathbf{E} \left[\mathbf{1}_{|B|/N \leq \beta_{\max}} \cdot \operatorname{tr} \left| \Pi_B \left(D_{\gamma}^{(\eta)} - d_{\eta} I \right) \Pi_B \right|^p \right]^{1/p} \leq N^{1/p} \varepsilon.$$

Proof. Write

$$\widehat{M}_{\gamma}^{(\eta)} - m_{\eta} I = H^{\otimes n} \left(\frac{1}{n} \sum_i P_i \right) H^{\otimes n}, \quad P_i = \sum_{s: s_i=0} |s_{[1,i-1]} \rangle \langle s_{[1,i-1]}| \otimes P_{i,s} \otimes |s_{[i+1,n]} \rangle \langle s_{[i+1,n]}|,$$

$$\text{and} \quad P_{i,s} = \frac{|g_{s,i} \rangle \langle g_{s,i}|}{\| |g_{s,i} \rangle \|_2^2 + \delta/N} - m_{\eta} I,$$

where $|g_{s,i} \rangle = \gamma_s |0\rangle + \gamma_{s+i} |1\rangle$.

Fix $i \in [n]$ and for convenience reorder the n tensor factors so that i is last. Then P_i is a direct sum of 2×2 matrices with operator norm $\mathcal{O}_{\delta}(1)$,

$$P_i = \sum_{s: s_i=0} |s_{[n] \setminus i} \rangle \langle s_{[n] \setminus i}| \otimes P_{i,s},$$

so $\|P_i\|_{\text{op}} = \|H^{\otimes n} P_i H^{\otimes n}\|_{\text{op}} = \mathcal{O}_\delta(1)$ and indeed $\|\widehat{M}_\gamma^{(\eta)} - m_\eta I\|_{\text{op}} = \mathcal{O}_\delta(1)$.

To see that $H^{\otimes n} P_i H^{\otimes n}$ is entrywise small, note that, schematically

$$\begin{aligned} H^{\otimes n} P_i H^{\otimes n} &= \sum_{s: s_i=0} \left(H^{\otimes n-1} |s_{[n]\setminus i}\rangle\langle s_{[n]\setminus i}| H^{\otimes n-1} \right) \otimes \left(H P_{i,s} H \right) \\ &= \frac{2}{N} \sum_{s: s_i=0} \left(\pm 1 \text{ sign matrix} \right) \otimes \left(H P_{i,s} H \right). \end{aligned}$$

Each 2×2 matrix in the family $\{P_{i,s}\}_s$ is independent, mean-zero, and bounded, so we see from this display that each entry of $H^{\otimes n} P_i H^{\otimes n}$ is an average of $N/2$ -many $\mathcal{O}_\delta(1)$ -bounded, centered random variables.

By Bernstein's inequality, this means there is a universal constant $c' = c'(\delta)$ such that for every fixed pair x, y ,

$$\Pr \left[|(H^{\otimes n} P_i H^{\otimes n})_{xy}| > \frac{1}{n^2} \right] \leq \exp \left(-c' \frac{N}{n^4} \right).$$

Taking a union bound over all $x, y \in \{0, 1\}^n$ and all $i \in [n]$, the event

$$E_M := \left\{ \max_{i \in [n]} \max_{x, y} |(H^{\otimes n} P_i H^{\otimes n})_{xy}| \leq \frac{1}{n^2} \right\}$$

holds with probability $1 - \exp(-N/\text{poly}(n))$. When E obtains, we get

$$\max_{x, y} \left| \left(\widehat{M}_\gamma^{(\eta)} - m_\eta I \right)_{xy} \right| \leq \frac{1}{n} \sum_i \max_{x, y} |(H^{\otimes n} P_i H^{\otimes n})_{xy}| \leq \frac{1}{n^2}$$

for all sufficiently large n .

The same argument applies to the centered dominator $D_\gamma^{(\delta)} - d_\eta I$. Indeed,

$$D_\gamma^{(\delta)} - d_\eta I = \frac{1}{n} \sum_{i=1}^n Q_i,$$

where Q_i is the physical-side conjugate of the Fourier-side direct sum over i -edges of the centered scalar block

$$\left(\frac{\delta/N}{\| |g_{s,i}\rangle \|_2^2 + \delta/N} - d_\eta \right) I_2.$$

In each Q_i these blocks are i.i.d., mean-zero, and bounded. Hence, on an event E_D analogous to E_M , also holding with probability $\exp(-N/\text{poly}(n))$,

$$\left\| D_\gamma^{(\delta)} - d_\eta I \right\|_{\text{op}} = \mathcal{O}_\eta(1) \quad \text{and} \quad \max_{x, y} \left| \left(D_\gamma^{(\delta)} - d_\eta I \right)_{xy} \right| \leq \frac{1}{n^2}$$

for all sufficiently large n .

To finish the argument, we apply [Theorem 10](#) to the deterministic matrices $\widehat{M}_\gamma^{(\eta)} - m_\eta I$ and $D_\gamma^{(\eta)} - d_\eta I$. The application is identical for each, so we explain the former. Let $E = E_M \cap E_D$ and let $C_\eta \leq \mathcal{O}_\eta(1)$ be such that $\|\widehat{M}_\gamma^{(\eta)}\|_{\text{op}} \leq C_\eta$ pointwise. Let ε be as in the statement of [Proposition 19](#) and put $\varepsilon' = \varepsilon/C_\eta$. Then according to [Theorem 10](#) there is a density threshold $\beta_{\max} = \beta_{\max}(\varepsilon'/2, 1)$ such that for all even $2 \leq p \leq n$,

$$\begin{aligned} \mathbf{E} \left[\mathbf{1}_{|B|/N \leq \beta_{\max}} \cdot \left\| \Pi_B (\widehat{M}_\gamma^{(\eta)} - m_\eta I) \Pi_B \right\|_{\text{op}}^p \right] &= \sum_{b=0}^{\lfloor \beta_{\max} N \rfloor} \mathbf{E} \left[\mathbf{1}_{|B|=b \wedge E} \cdot \left\| \Pi_B (\widehat{M}_\gamma^{(\eta)} - m_\eta I) \Pi_B \right\|_{\text{op}}^p \right] \\ &\quad + \mathbf{E} \left[\mathbf{1}_{|B|/N \leq \beta_{\max} \wedge E^c} \cdot \left\| \Pi_B (\widehat{M}_\gamma^{(\eta)} - m_\eta I) \Pi_B \right\|_{\text{op}}^p \right] \\ &\leq (\varepsilon/2)^p + \mathcal{O}_\eta(1)^p \cdot \exp(-N/\text{poly}(n)) \\ &\leq \varepsilon^p. \end{aligned}$$

for n large enough. We conclude that for all even $2 \leq p \leq n$,

$$\begin{aligned} \mathbf{E} \left[\mathbf{1}_{|B|/N \leq \beta_{\max}} \cdot \text{tr} \left| \Pi_B (\widehat{M}_{\gamma}^{(\eta)} - m_{\eta} I) \Pi_B \right|^p \right]^{1/p} \\ \leq N^{1/p} \mathbf{E} \left[\mathbf{1}_{|B|/N \leq \beta_{\max}} \left\| \Pi_B (\widehat{M}_{\gamma}^{(\eta)} - m_{\eta} I) \Pi_B \right\|_{\text{op}}^p \right]^{1/p} \leq N^{1/p} \varepsilon, \end{aligned}$$

as desired. $\square$

Step 4: Combine bounds. It remains to combine [Proposition 18](#) and [Proposition 19](#) to prove the sufficient condition ([Proposition 17](#)) for the BB bound, [Equation \(25\)](#).

Proof of [Proposition 17](#). Fix $\varepsilon > 0$ and choose $\eta > 0$ sufficiently small so that

$$d_{\eta} \leq \frac{\varepsilon}{2}.$$

Recall from [\(29\)](#) that $m_{\eta} \leq m_{\eta} + d_{\eta}/2 = 1/2$.

Let C_{η} be the constant from [Proposition 18](#). Choose a constant $c = c(\eta, \varepsilon) \in (0, 1)$ sufficiently small that

$$2C_{\eta} \exp\left(-\frac{3 \log 2}{2c}\right) \leq \frac{\varepsilon}{8} \exp\left(-\frac{5 \log 2}{4c}\right).$$

Apply [Proposition 19](#) with regularization parameter η and paving target

$$\frac{\varepsilon}{8} \exp\left(-\frac{5 \log 2}{4c}\right),$$

and let $\beta_{\max} > 0$ be the resulting density threshold.

By [Proposition 16](#), there are deterministic quantities $r_T \rightarrow 0$ as $T \rightarrow \infty$ such that, with probability $1 - \exp(-\Omega(n))$,

$$\|\mu\|_{3/2}^{3/2} \leq N^{-1/2+o(1)}, \quad \text{Bias}(\mu) \leq N^{-1/2+o(1)},$$

and

$$\frac{|B|}{N} + \max_y \mu(B + y) \leq r_T.$$

Choose T sufficiently large that

$$r_T \leq \beta_{\max}$$

and

$$C_{\eta} \sqrt{r_T} \leq \frac{\varepsilon}{8} \exp\left(-\frac{5 \log 2}{4c}\right).$$

Let E denote the event on which these profile bounds hold.

For each n , let p be the largest even integer satisfying $p \leq cn$. For all sufficiently large n , we have $8 \leq p \leq n$. Moreover, on event E ,

$$\text{Bias}(\mu)^{2/p} \leq \exp\left(-\frac{\log 2 - o(1)}{c}\right) \leq \frac{1}{2}.$$

Also,

$$\|\mu\|_{3/2}^{1/2} = \left(\|\mu\|_{3/2}^{3/2}\right)^{1/3} \leq N^{-1/6+o(1)}.$$

Thus, after increasing n if necessary,

$$C_{\eta} p^3 \|\mu\|_{3/2}^{1/2} \leq \frac{\varepsilon}{8} \exp\left(-\frac{5 \log 2}{4c}\right)$$

and

$$C_\eta \text{Bias}(\mu)^{3/p} \leq \frac{\varepsilon}{8} \exp\left(-\frac{5 \log 2}{4c}\right).$$

The following argument applies identically to the centered regularized M -operator and to the centered dominator. Write $W_{\hat{f}}$ and W_γ for either of the pairs

$$W_{\hat{f}} = \Pi_B \left(\widehat{M}_{\hat{f}}^{(\eta)} - m_\eta I \right) \Pi_B, \quad W_\gamma = \Pi_B \left(\widehat{M}_\gamma^{(\eta)} - m_\eta I \right) \Pi_B,$$

or

$$W_{\hat{f}} = \Pi_B \left(D_{\hat{f}}^{(\eta)} - d_\eta I \right) \Pi_B, \quad W_\gamma = \Pi_B \left(D_\gamma^{(\eta)} - d_\eta I \right) \Pi_B.$$

Conditional on $\mu \in E$, [Proposition 18](#) gives

$$\begin{aligned} \left(\mathbf{E}_\omega \text{tr} |W_{\hat{f}}|^p \right)^{1/p} &\leq \left(\mathbf{E}_\gamma \text{tr} |W_\gamma|^p \right)^{1/p} \\ &\quad + \frac{3\varepsilon}{8} N^{1/p} \exp\left(-\frac{5 \log 2}{4c}\right). \end{aligned}$$

Taking the L_p -norm over μ on E and applying Minkowski's inequality,

$$\begin{aligned} \left(\mathbf{E}_\mu \mathbf{1}_E \mathbf{E}_\omega \text{tr} |W_{\hat{f}}|^p \right)^{1/p} &\leq \left(\mathbf{E}_\mu \mathbf{1}_E \mathbf{E}_\gamma \text{tr} |W_\gamma|^p \right)^{1/p} \\ &\quad + \frac{3\varepsilon}{8} N^{1/p} \exp\left(-\frac{5 \log 2}{4c}\right). \end{aligned}$$

Since E implies $|B|/N \leq \beta_{\max}$, [Proposition 19](#) gives

$$\left(\mathbf{E}_\mu \mathbf{1}_E \mathbf{E}_\gamma \text{tr} |W_\gamma|^p \right)^{1/p} \leq \frac{\varepsilon}{8} N^{1/p} \exp\left(-\frac{5 \log 2}{4c}\right).$$

Consequently,

$$\left(\mathbf{E}_\mu \mathbf{1}_E \mathbf{E}_\omega \text{tr} |W_{\hat{f}}|^p \right)^{1/p} \leq \frac{\varepsilon}{2} N^{1/p} \exp\left(-\frac{5 \log 2}{4c}\right).$$

Markov's inequality therefore gives

$$\begin{aligned} \Pr \left[E \wedge \|W_{\hat{f}}\|_{\text{op}} \geq \frac{\varepsilon}{2} \right] &\leq \left(\frac{2}{\varepsilon} \right)^p \mathbf{E}_\mu \mathbf{1}_E \mathbf{E}_\omega \text{tr} |W_{\hat{f}}|^p \\ &\leq N \exp\left(-\frac{5 \log 2}{4c} p\right) \\ &= \exp\left(-\frac{\log 2}{4} n + O_c(1)\right). \end{aligned}$$

Since $\Pr[E^c] \leq \exp(-\Omega(n))$, applying the estimate to both centered objects and taking a union bound shows that, with probability $1 - \exp(-\Omega(n))$,

$$\left\| \Pi_B \left(\widehat{M}_{\hat{f}}^{(\eta)} - m_\eta I \right) \Pi_B \right\|_{\text{op}} \leq \frac{\varepsilon}{2} \quad \text{and} \quad \left\| \Pi_B \left(D_{\hat{f}}^{(\eta)} - d_\eta I \right) \Pi_B \right\|_{\text{op}} \leq \frac{\varepsilon}{2}.$$

Therefore,

$$\begin{aligned} \left\| \Pi_B \widehat{M}_{\hat{f}}^{(\eta)} \Pi_B \right\|_{\text{op}} &\leq m_\eta + \frac{\varepsilon}{2} \\ &\leq \frac{1}{2} + \varepsilon, \end{aligned}$$

and

$$\begin{aligned} \left\| \Pi_B D_{\hat{f}}^{(\eta)} \Pi_B \right\|_{\text{op}} &\leq d_\eta + \frac{\varepsilon}{2} \\ &\leq \varepsilon. \end{aligned}$$

This proves the proposition. □

7.3 Proof of Equation (26), the AB , BA bounds

The proof of Equation (26) goes much the same way as the BB bound, except we analyze the Hermitian square. We describe it only briefly. For notational convenience define

$$R_{\hat{f}}^{(\eta)} := \widehat{M}_{\hat{f}}^{(\eta)} - m_{\eta}I.$$

Since $A \cap B = \emptyset$,

$$\Pi_A \widehat{M}_{\hat{f}}^{(\eta)} \Pi_B = \Pi_A R_{\hat{f}}^{(\eta)} \Pi_B.$$

Consequently,

$$\left\| \Pi_A \widehat{M}_{\hat{f}}^{(\eta)} \Pi_B \right\|_{\text{op}}^2 = \left\| \Pi_B R_{\hat{f}}^{(\eta)} \Pi_A R_{\hat{f}}^{(\eta)} \Pi_B \right\|_{\text{op}} \leq \left\| \Pi_B \left(R_{\hat{f}}^{(\eta)} \right)^2 \Pi_B \right\|_{\text{op}}. \quad (56)$$

Define also

$$R_{\gamma}^{(\eta)} := \widehat{M}_{\gamma}^{(\eta)} - m_{\eta}I.$$

Lemma 7 (Gaussian square mean). *There is a scalar $b_{n,\eta}$ such that*

$$\mathbf{E}_{\gamma} \left(R_{\gamma}^{(\eta)} \right)^2 = b_{n,\eta} I.$$

Moreover,

$$\lim_{\eta \downarrow 0} \lim_{n \rightarrow \infty} b_{n,\eta} = b,$$

where

$$b = \frac{\pi^2}{3} - \frac{13}{4} < \frac{1}{16}.$$

Proof. Write

$$R_{\gamma}^{(\eta)} = H^{\otimes n} \left(\frac{1}{n} \sum_{i=1}^n P_i^{(\eta)} \right) H^{\otimes n},$$

where

$$P_i^{(\eta)} = \bigoplus_{s: s_i=0} P_{i,s}^{(\eta)}$$

and

$$P_{i,s}^{(\eta)} := \frac{|g_{s,i}\rangle\langle g_{s,i}|}{\|g_{s,i}\|_2^2 + \eta/N} - m_{\eta}I_2, \quad |g_{s,i}\rangle := \gamma_s |0\rangle + \gamma_{s+e_i} |1\rangle.$$

The law of γ is invariant under translations of the cube and under independent phase rotations of its coordinates. It follows that, for some scalars a_{η} and b_{η} ,

$$\mathbf{E} \left(P_i^{(\eta)} \right)^2 = a_{\eta} I$$

and, whenever $i \neq j$,

$$\mathbf{E} P_i^{(\eta)} P_j^{(\eta)} = b_{\eta} I.$$

Moreover, $\|P_i^{(\eta)}\|_{\text{op}} \leq 1$, so $0 \leq a_{\eta} \leq 1$.

To identify b_{η} , fix distinct i, j and a vertex s . The i -edge and j -edge through s meet only at s , and hence the s -th diagonal entry of $P_i^{(\eta)} P_j^{(\eta)}$ is the product of the corresponding diagonal entries. Thus, if X, Y, Z are independent $\text{Exp}(1)$ random variables,

$$b_{\eta} = \mathbf{E} \left[\left(\frac{X}{X+Y+\eta} - m_{\eta} \right) \left(\frac{X}{X+Z+\eta} - m_{\eta} \right) \right],$$

where

$$m_\eta = \mathbf{E} \frac{X}{X + Y + \eta}.$$

Consequently,

$$\mathbf{E}_\gamma \left(R_\gamma^{(\eta)} \right)^2 = H^{\otimes n} \mathbf{E} \left[\left(\frac{1}{n} \sum_i P_i^{(\eta)} \right)^2 \right] H^{\otimes n} = \left(\frac{a_\eta}{n} + \frac{n-1}{n} b_\eta \right) I.$$

We may therefore take

$$b_{n,\eta} := \frac{a_\eta}{n} + \frac{n-1}{n} b_\eta.$$

Since a_η is bounded,

$$\lim_{n \rightarrow \infty} b_{n,\eta} = b_\eta.$$

Also, by dominated convergence,

$$m_\eta \rightarrow \frac{1}{2}$$

and

$$b_\eta \rightarrow \mathbf{E} \left[\left(\frac{X}{X+Y} - \frac{1}{2} \right) \left(\frac{X}{X+Z} - \frac{1}{2} \right) \right] = \mathbf{E} \frac{X^2}{(X+Y)(X+Z)} - \frac{1}{4}.$$

The normalized vector

$$\frac{(X, Y, Z)}{X + Y + Z}$$

is uniform on the two-dimensional simplex. Therefore,

$$\begin{aligned} \mathbf{E} \frac{X^2}{(X+Y)(X+Z)} &= 2 \int_0^1 \int_0^{1-x} \frac{x^2}{(x+y)(1-y)} dy dx \\ &= -4 \int_0^1 \frac{x^2 \log x}{1+x} dx \\ &= 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+3)^2} \\ &= \frac{\pi^2}{3} - 3. \end{aligned}$$

It follows that

$$b = \frac{\pi^2}{3} - 3 - \frac{1}{4} = \frac{\pi^2}{3} - \frac{13}{4} < \frac{1}{16}. \quad \square$$

We claim that, for every fixed $\varepsilon > 0$, after choosing T sufficiently large, with probability $1 - \exp(-\Omega(n))$,

$$\left\| \Pi_B \left(\left(R_{\hat{f}}^{(\eta)} \right)^2 - b_{n,\eta} I \right) \Pi_B \right\|_{\text{op}} \leq \varepsilon. \quad (57)$$

The proof is the same root-level Gaussian-transfer and paving argument used in the BB case. Indeed, since $\|R^{(\eta)}\|_{\text{op}} \leq 1$, the product rule and [Lemma 5](#) give the same physical-coordinate derivative bounds for $(R^{(\eta)})^2$. Its first and second raw Fourier-coordinate derivatives likewise have the same $C_\eta \sqrt{N}$ and $C_\eta N$ scales as before. After expanding the product rule, the covariance-interpolation terms have the same T_1 , interior- T_2 , and endpoint- T_2 forms as in the BB proof; the additional factors of $R^{(\eta)}$ are contractions and are absorbed into the adjacent Schatten factors. The centered Gaussian square is uniformly bounded and satisfies the same entrywise estimate required by [Theorem 10](#). Thus the proofs of [Proposition 18](#) and [Proposition 19](#) apply without further change and yield (57).

Combining (56) and (57), we obtain

$$\left\| \Pi_A \widehat{M}_{\hat{f}}^{(\eta)} \Pi_B \right\|_{\text{op}} \leq \sqrt{b_{n,\eta}} + \varepsilon.$$

Choosing η sufficiently small, then n sufficiently large, and taking the tolerance in (57) sufficiently small gives

$$\left\| \Pi_A \widehat{M}_{\hat{f}}^{(\eta)} \Pi_B \right\|_{\text{op}} \leq \frac{1}{4} + \frac{\varepsilon}{2}.$$

It remains only to remove the regularization. Put

$$E_{\hat{f}}^{(\eta)} := \widehat{M}_{\hat{f}} - \widehat{M}_{\hat{f}}^{(\eta)}.$$

Since

$$0 \preceq E_{\hat{f}}^{(\eta)} \preceq D_{\hat{f}}^{(\eta)} \quad \text{and} \quad E_{\hat{f}}^{(\eta)} \preceq I,$$

we have

$$\begin{aligned} \left\| \Pi_A E_{\hat{f}}^{(\eta)} \Pi_B \right\|_{\text{op}}^2 &= \left\| \Pi_B E_{\hat{f}}^{(\eta)} \Pi_A E_{\hat{f}}^{(\eta)} \Pi_B \right\|_{\text{op}} \\ &\leq \left\| \Pi_B \left(E_{\hat{f}}^{(\eta)} \right)^2 \Pi_B \right\|_{\text{op}} \\ &\leq \left\| \Pi_B D_{\hat{f}}^{(\eta)} \Pi_B \right\|_{\text{op}}. \end{aligned}$$

Applying Proposition 17 with a sufficiently small error parameter therefore gives

$$\left\| \Pi_A E_{\hat{f}}^{(\eta)} \Pi_B \right\|_{\text{op}} \leq \frac{\varepsilon}{2}$$

with probability $1 - \exp(-\Omega(n))$. Hence

$$\left\| \Pi_A \widehat{M}_{\hat{f}} \Pi_B \right\|_{\text{op}} \leq \frac{1}{4} + \varepsilon,$$

which proves Equation (26).

8 Finishing the proof

Proof of Theorem 4. Fix $\varepsilon < 1/20$ and choose $T > 0$ large enough to satisfy the conditions in Theorem 9; *i.e.*, so that the errors in the AB - and BB -bounds are at most ε . Then choose $\kappa > 0$ small enough that

$$2\sqrt{\frac{CT^5\kappa}{\varepsilon^3}} < \frac{1}{3} \quad \text{and} \quad \kappa < \frac{1}{4} - 5\varepsilon,$$

where C is the constant in Theorem 8.

If $\mathbf{Avg}_f[g] \leq 1 - \kappa$, we are done, so assume $\mathbf{Avg}_f[g] > 1 - \kappa$, *i.e.*,

$$\mathbf{Inf}_\mu[q] < \kappa n.$$

It will be convenient to work with a shifted q (equiv. g) so that its A -part is centered with respect to μ . To that end, let $A = B^c$, and define

$$\alpha := \frac{\langle \mathbf{1}, q\mathbf{1}_A \rangle_\mu}{\langle \mathbf{1}, \mathbf{1}_A \rangle_\mu}.$$

Then define the shifted versions of g and q by:

$$\begin{aligned} g^\perp &:= g - \alpha f, & g_A^\perp &:= g^\perp \mathbf{1}_A, & g_B^\perp &:= g^\perp \mathbf{1}_B \\ \text{and } q^\perp &:= q - \alpha \mathbf{1}, & q_A^\perp &:= q^\perp \mathbf{1}_A, & q_B^\perp &:= q^\perp \mathbf{1}_B \end{aligned}$$

Note that since $g = fq$, $\langle \mathbf{1}, q \rangle_\mu = 0$, and $\|q\|_{L^2(\mu)} = 1$, we have

$$\langle f, g^\perp \rangle = \langle \mathbf{1}, q^\perp \rangle_\mu = -\alpha \quad \text{and} \quad \|g^\perp\|_2^2 = \|q^\perp\|_{L^2(\mu)}^2 = 1 + |\alpha|^2.$$

Moreover, the definition of α gives

$$\langle f, g_A^\perp \rangle = \mathbf{E}_\mu q_A^\perp = 0.$$

Since f is a $+1$ eigenvector of $\widehat{M}_{\widehat{f}}$,

$$\begin{aligned} \mathbf{Avg}_{\widehat{f}}[\widehat{g}] &= \langle g, \widehat{M}_{\widehat{f}} g \rangle \\ &= \langle g^\perp + \alpha f, \widehat{M}_{\widehat{f}}(g^\perp + \alpha f) \rangle \\ &= \langle g^\perp, \widehat{M}_{\widehat{f}} g^\perp \rangle + 2\Re(\bar{\alpha} \langle f, g^\perp \rangle) + |\alpha|^2 \\ &= \langle g^\perp, \widehat{M}_{\widehat{f}} g^\perp \rangle - |\alpha|^2. \end{aligned}$$

Decomposing $g^\perp$, we get

$$\langle g^\perp, \widehat{M}_{\widehat{f}} g^\perp \rangle = \langle g_A^\perp, \widehat{M}_{\widehat{f}} g_A^\perp \rangle + 2\Re \langle g_A^\perp, \widehat{M}_{\widehat{f}} g_B^\perp \rangle + \langle g_B^\perp, \widehat{M}_{\widehat{f}} g_B^\perp \rangle. \quad (58)$$

We bound these three terms separately. Note that our choice of α ensures that $q_A^\perp$ is centered with respect to μ , as required for the AA -part argument, while one appreciates that the AB - and BB -operator bounds of [Theorem 9](#) require no centeredness.

Indeed, by [Theorem 9](#) and our choice of T , with probability $1 - \exp(-\Omega(n))$ we have

$$\left| \langle g_A^\perp, \widehat{M}_{\widehat{f}} g_B^\perp \rangle \right| \leq \left(\frac{1}{4} + \varepsilon \right) \|g_A^\perp\|_2 \|g_B^\perp\|_2 \quad \text{and} \quad \langle g_B^\perp, \widehat{M}_{\widehat{f}} g_B^\perp \rangle \leq \left(\frac{1}{2} + \varepsilon \right) \|g_B^\perp\|_2^2.$$

AA-part. For the AA part, $\langle g_A^\perp, \widehat{M}_{\widehat{f}} g_A^\perp \rangle$, we pass to the q picture. Put

$$\widetilde{M} := \text{Diag}(1/f) \widehat{M}_{\widehat{f}} \text{Diag}(f).$$

This is well-defined with probability one. The map $u \mapsto fu$ is an isometry from $L^2(\mu)$ to ℓ_2 , and

$$\langle u, \widetilde{M}v \rangle_\mu = \langle fu, \widehat{M}_{\widehat{f}}fv \rangle.$$

Consequently, $\widetilde{M}$ is self-adjoint and contractive on $L^2(\mu)$, and

$$\langle g_A^\perp, \widehat{M}_{\widehat{f}} g_A^\perp \rangle = \langle q_A^\perp, \widetilde{M} q_A^\perp \rangle_\mu.$$

On the internal edges of A , the functions $q_A^\perp$ and $q - \alpha \mathbf{1}$ agree. Therefore

$$\begin{aligned} \mathbf{Inf}_\mu^{\text{int}(A)}[q_A^\perp] &= \mathbf{Inf}_\mu^{\text{int}(A)}[q - \alpha \mathbf{1}] \\ &\leq \mathbf{Inf}_\mu[q - \alpha \mathbf{1}] \\ &= \mathbf{Inf}_\mu[q] \\ &\leq \kappa n. \end{aligned}$$

We claim that, with high probability,

$$\langle g_A^\perp, \widehat{M}_{\widehat{f}} g_A^\perp \rangle = \langle q_A^\perp, \widetilde{M} q_A^\perp \rangle_\mu \leq \left(\frac{1}{2} + 2\varepsilon + o(1) \right) \|q_A^\perp\|_{L^2(\mu)}^2 + \varepsilon. \quad (59)$$

This follows by considering two cases.

Case I: Small norm. If $\|q_A^\perp\|_{L^2(\mu)}^2 \leq \varepsilon$, then contractivity of $\widetilde{M}$ gives

$$\langle q_A^\perp, \widetilde{M} q_A^\perp \rangle_\mu \leq \|q_A^\perp\|_{L^2(\mu)}^2 \leq \varepsilon,$$

so (59) holds.

Case II: Large norm. Suppose instead that $\|q_A^\perp\|_{L^2(\mu)}^2 > \varepsilon$. Then

$$\tilde{q}_A^\perp := \frac{q_A^\perp}{\|q_A^\perp\|_{L^2(\mu)}}$$

satisfies

$$\mathbf{E}_\mu \tilde{q}_A^\perp = 0, \quad \|\tilde{q}_A^\perp\|_{L^2(\mu)} = 1, \quad \text{and} \quad \mathbf{Inf}_\mu^{\text{int}(A)}[\tilde{q}_A^\perp] \leq \frac{\kappa}{\varepsilon} n.$$

On the high-probability event over μ on which [Corollary 6](#) and [Theorem 5](#) both hold, let

$$\mathcal{Q}_{\kappa/\varepsilon, A}^{(\varepsilon/4)}$$

be the compatible flat net supplied by [Corollary 6](#). Every $v \in \mathcal{Q}_{\kappa/\varepsilon, A}^{(\varepsilon/4)}$ is μ -compatible and satisfies

$$\max_x \mu(x) |v(x)|^2 \leq \|\mu\|_\infty^{1/3}.$$

Moreover, by the definition of $\widetilde{M}$, $\langle v, \widetilde{M} v \rangle_\mu = \mathbf{Avg}_{\widehat{f}}[\widehat{f}v]$. We may therefore apply [Theorem 5](#) to every point of the net. Taking the deviation parameter there to be $\varepsilon/2$ and applying a union bound gives

$$\begin{aligned} & \Pr_{f \sim F_\mu} \left[\exists v \in \mathcal{Q}_{\kappa/\varepsilon, A}^{(\varepsilon/4)} \text{ such that } \langle v, \widetilde{M} v \rangle_\mu > \frac{1}{2} + \frac{\varepsilon}{2} + o(1) \right] \\ & \leq \exp \left(N^{H(CT^5 \kappa / \varepsilon^3) + o(1)} \log \left(\frac{C}{\varepsilon} \right) - c\varepsilon^6 N^{1/3 - o(1)} \right). \end{aligned}$$

Here and below, the universal constants absorb the fixed factors arising from using κ/ε as the energy threshold and $\varepsilon/4$ as the net radius. By our choice of κ ,

$$H \left(\frac{CT^5 \kappa}{\varepsilon^3} \right) < \frac{1}{3}.$$

Consequently, the preceding probability is

$$\exp \left(-N^{1/3 - o(1)} \right),$$

and in particular is at most $\exp(-\Omega(n))$.

Condition on the complementary event. Since

$$\tilde{q}_A^\perp \in \mathcal{Q}_{\kappa/\varepsilon, A},$$

we may choose

$$v \in \mathcal{Q}_{\kappa/\varepsilon, A}^{(\varepsilon/4)}$$

such that

$$\|\tilde{q}_A^\perp - v\|_{L^2(\mu)} \leq \frac{\varepsilon}{4}.$$

Since $\tilde{q}_A^\perp$ and v both have unit $L^2(\mu)$ norm and $\|\widetilde{M}\|_{\text{op}, L^2(\mu)} \leq 1$,

$$\begin{aligned} \left\langle \tilde{q}_A^\perp, \widetilde{M} \tilde{q}_A^\perp \right\rangle_\mu &\leq \langle v, \widetilde{M} v \rangle_\mu + \left| \left\langle \tilde{q}_A^\perp - v, \widetilde{M} \tilde{q}_A^\perp \right\rangle_\mu \right| + \left| \left\langle v, \widetilde{M} (\tilde{q}_A^\perp - v) \right\rangle_\mu \right| \\ &\leq \frac{1}{2} + \frac{\varepsilon}{2} + o(1) + 2\|\tilde{q}_A^\perp - v\|_{L^2(\mu)} \\ &\leq \frac{1}{2} + \varepsilon + o(1). \end{aligned}$$

Multiplying by

$$\|q_A^\perp\|_{L^2(\mu)}^2$$

gives

$$\langle q_A^\perp, \widetilde{M} q_A^\perp \rangle_\mu \leq \left(\frac{1}{2} + \varepsilon + o(1) \right) \|q_A^\perp\|_{L^2(\mu)}^2,$$

which is stronger than (59).

Combining the bounds. Intersecting the good events for the AA , AB , and BB bounds and substituting into (58), we obtain, with probability $1 - \exp(-\Omega(n))$,

$$\begin{aligned} \langle g^\perp, \widehat{M}_{\widehat{f}} g^\perp \rangle &\leq (\|g_A^\perp\|_2 \quad \|g_B^\perp\|_2) \begin{pmatrix} \frac{1}{2} + 2\varepsilon + o(1) & \frac{1}{4} + \varepsilon \\ \frac{1}{4} + \varepsilon & \frac{1}{2} + \varepsilon \end{pmatrix} \begin{pmatrix} \|g_A^\perp\|_2 \\ \|g_B^\perp\|_2 \end{pmatrix} + \varepsilon \\ &\leq \left(\frac{3}{4} + 3\varepsilon + o(1) \right) (\|g_A^\perp\|_2^2 + \|g_B^\perp\|_2^2) + \varepsilon \\ &= \left(\frac{3}{4} + 3\varepsilon + o(1) \right) (1 + |\alpha|^2) + \varepsilon. \end{aligned}$$

Here the maximum eigenvalue of the displayed 2×2 matrix is at most its largest row sum, namely $\frac{3}{4} + 3\varepsilon + o(1)$.

Returning to the earlier identity,

$$\begin{aligned} \mathbf{Avg}_{\widehat{f}}[\widehat{g}] &= \langle g^\perp, \widehat{M}_{\widehat{f}} g^\perp \rangle - |\alpha|^2 \\ &\leq \frac{3}{4} + 4\varepsilon + o(1) - \left(\frac{1}{4} - 3\varepsilon - o(1) \right) |\alpha|^2 \\ &\leq \frac{3}{4} + 5\varepsilon \end{aligned}$$

for all sufficiently large n . By our choice of κ , this means $\mathbf{Avg}_{\widehat{f}}[\widehat{g}] < 1 - \kappa$.

We have therefore shown that, with probability $1 - \exp(-\Omega(n))$ over f , for every admissible g ,

$$\min\{\mathbf{Avg}_f[g], \mathbf{Avg}_{\widehat{f}}[\widehat{g}]\} \leq 1 - \kappa.$$

This proves [Theorem 4](#). □

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A Notational glossary

F_μ	The set of $f : \{0, 1\}^n \rightarrow \mathbb{C}$ such that $\mu_f = \mu$, where $\mu_f(x) = f(x) ^2$.
η	Dimensionless smoothing parameter used in the regularization.
δ	Dimensional smoothing parameter, related to η by $\delta = \eta/N$.
β	Density of the bad set $B \subseteq \{0, 1\}^n$.
$\beta_{\max}$	Density threshold at which Tropp's random-paving result applies.
$\text{Bias}(\mu)$	Largest magnitude among $\widehat{\mu}(s)$, $s \neq 0^n$.
D	Fréchet derivative.
m_η	Centering constant used in the Gaussianization of $\widehat{M}_{\widehat{f}}$.
d_η	Centering constant used in the Gaussianization of $D_{\widehat{f}}$.

B Large covering number but small Inf_μ

Proposition 20. *For every $\kappa > 0$, with probability at least $1 - \exp(-\Omega_\kappa(2^n))$, there exists a subspace V of functions on $\{0, 1\}^n$ of dimension $\Omega_\kappa(2^n)$ such that every $q \in V$, $\|q\|_{L^2(\mu)} = 1$, satisfies*

$$\text{Inf}_\mu[f] \leq \kappa n.$$

The idea is to construct a subspace of functions with basis $q_x := \mathbf{1}_{\{x\}}/\sqrt{\mu(x)}$ for x with $\mu(x)$ large yet $\mu(y)$ small for all neighbors $y \sim x$. Because the smaller weight on an edge governs the conductance (edge weight), $\text{Inf}_\mu(q_x)$ will stay small for these x .

Proof. It will be convenient to work with the Exponential model for the Porter–Thomas distribution. Write

$$\mu(x) = \frac{Z_x}{\sum_z Z_z},$$

where the Z_x , $x \in \{0, 1\}^n$ are independent $\text{Exp}(1)$ random variables.

Let $E, O \subset \{0, 1\}^n$ be all even and odd strings respectively. For $x \in E$, set

$$R_x = \sum_{y \sim x} Z_y.$$

Since every $y \in O$ has exactly n neighbors in E ,

$$\sum_{x \in E} R_x = n \sum_{y \in O} Z_y.$$

By a Chernoff bound,

$$\Pr\left(\sum_{y \in O} Z_y > \frac{3N}{4}\right) \leq \exp(-cN).$$

On the complementary event, the number of vertices $x \in E$ for which $R_x > 2n$ is at most

$$\frac{1}{2n} \sum_{x \in E} R_x = \frac{1}{2} \sum_{y \in O} Z_y \leq \frac{3N}{8}.$$

Since $|E| = N/2$, there are therefore at least $N/8$ vertices $x \in E$ satisfying

$$R_x \leq 2n.$$

Condition on the variables $(Z_y)_{y \in O}$. For the vertices satisfying $R_x \leq 2n$, the events

$$Z_x \geq \frac{2}{\kappa}$$

are independent. Define

$$p_\kappa = \Pr\left(Z \geq \frac{2}{\kappa}\right) > 0$$

for a fresh $Z \sim \text{Exp}(1)$. Another Chernoff bound shows that, except with probability $\exp(-\Omega_\kappa(N))$, at least $p_\kappa N/16$ vertices satisfy both

$$R_x \leq 2n \quad \text{and} \quad Z_x \geq \frac{2}{\kappa}. \quad (60)$$

Let $S \subseteq E$ be the set of such vertices. For every $x \in S$,

$$\sum_{y \sim x} \frac{\mu(y)}{\mu(x) + \mu(y)} = \sum_{y \sim x} \frac{Z_y}{Z_x + Z_y} \leq \frac{R_x}{Z_x} \leq \kappa n.$$

Now let

$$V = \{q : \text{supp}(q) \subseteq S\}.$$

Then

$$\dim V = |S| \geq \frac{p_\kappa}{16} N.$$

Moreover, since S is contained in one parity class, it is an independent set. Hence, for every $q \in V$, conditioned on the event in (60), we get

$$\text{Inf}_\mu[q] = \sum_{x \in S} \mu(x) |q(x)|^2 \sum_{y \sim x} \frac{\mu(y)}{\mu(x) + \mu(y)} \leq \kappa n \sum_{x \in S} \mu(x) |q(x)|^2 \leq \kappa n \|f\|_{L^2(\mu)}^2 = \kappa n.$$

as desired. $\square$

C Relating Avg_f to Inf_μ

Here we provide a proof of the identity

$$\mathbf{Avg}_f[g] = 1 - \frac{1}{n} \mathbf{Inf}_{\mu_f} \left[\frac{g}{f} \right].$$

We first use the following elementary identity. For any $a, b \geq 0$ and $u, v \in \mathbb{C}$,

$$\frac{|au + bv|^2}{a + b} = a|u|^2 + b|v|^2 - \frac{ab}{a + b} |u - v|^2.$$

Indeed,

$$|au + bv|^2 = a^2|u|^2 + b^2|v|^2 + 2ab \text{Re}(u\bar{v}),$$

and

$$\begin{aligned} (a + b)(a|u|^2 + b|v|^2) - ab|u - v|^2 &= (a^2 + ab)|u|^2 + (ab + b^2)|v|^2 - ab(|u|^2 + |v|^2 - 2\text{Re}(u\bar{v})) \\ &= a^2|u|^2 + b^2|v|^2 + 2ab \text{Re}(u\bar{v}) \\ &= |au + bv|^2. \end{aligned}$$

Dividing by $a + b$ gives the claimed identity.

Let $q = g/f$ (here we are assuming f is nonzero everywhere, which happens with probability 1 for a Haar-random f). We now apply the above identity with

$$a = |f(x)|^2, \quad b = |f(x+i)|^2, \quad u = q(x), \quad v = q(x+i).$$

This gives

$$\begin{aligned} \frac{||f(x)|^2 q(x) + |f(x+i)|^2 q(x+i)|^2}{|f(x)|^2 + |f(x+i)|^2} &= |f(x)|^2 |q(x)|^2 + |f(x+i)|^2 |q(x+i)|^2 \\ &\quad - \frac{|f(x)|^2 |f(x+i)|^2}{|f(x)|^2 + |f(x+i)|^2} |q(x) - q(x+i)|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{Avg}_f[g] &= \frac{1}{2n} \sum_i \sum_x \frac{|\overline{f(x)}g(x) + \overline{f(x+i)}g(x+i)|^2}{|f(x)|^2 + |f(x+i)|^2} \\ &= \frac{1}{2n} \sum_i \sum_x \frac{||f(x)|^2 q(x) + |f(x+i)|^2 q(x+i)|^2}{|f(x)|^2 + |f(x+i)|^2} \\ &= \frac{1}{2n} \sum_{i,x} \left(|f(x)|^2 |q(x)|^2 + |f(x+i)|^2 |q(x+i)|^2 \right) \\ &\quad - \frac{1}{2n} \sum_{i,x} \frac{|f(x)|^2 |f(x+i)|^2}{|f(x)|^2 + |f(x+i)|^2} |q(x) - q(x+i)|^2. \end{aligned}$$

Now, notice that

$$\begin{aligned} \frac{1}{2n} \sum_{i,x} \left(|f(x)|^2 |q(x)|^2 + |f(x+i)|^2 |q(x+i)|^2 \right) &= \frac{1}{2n} \sum_i 2 \sum_x |f(x)|^2 |q(x)|^2 \\ &= \sum_x |f(x)|^2 |q(x)|^2 \\ &= \sum_x |g(x)|^2 = 1. \end{aligned}$$

Substituting this into the previous expression, we get

$$\begin{aligned} &\frac{1}{2n} \sum_i \sum_x \frac{||f(x)|^2 q(x) + |f(x+i)|^2 q(x+i)|^2}{|f(x)|^2 + |f(x+i)|^2} \\ &= 1 - \frac{1}{2n} \sum_i \sum_x \frac{|f(x)|^2 |f(x+i)|^2}{|f(x)|^2 + |f(x+i)|^2} |q(x) - q(x+i)|^2 \\ &= 1 - \frac{1}{n} \mathbf{Inf}_{\mu_f} \left[\frac{g}{f} \right], \end{aligned}$$

as desired.