

Sharp Bounds on Ground State Energy of the SYK Model

Arpon Basu
arpon.basu@princeton.edu
Princeton University

Pravesh K. Kothari
kothari@cs.princeton.edu
Princeton University

Siddhant Midha
siddhantm@princeton.edu
Princeton University

July 30, 2026

Abstract

We study the Sachdev-Ye-Kitaev (SYK) Hamiltonian H_{SYK} on n Majorana modes with k -body interactions, and prove that $\mathbb{E} \|H_{\text{SYK}}\|_{\text{op}} = (1 - o(1)) \cdot \sqrt{2n}/k$ for super-constant $k \leq o(\sqrt{n})$, where the expectation is over the disorder variables in the Hamiltonian. This confirms the predictions due to [GGJV18] and answers a question posed in [FTW19]. Our results extend to the sparse SYK Hamiltonian. As a corollary, we obtain that the dissipative quantum algorithm of [BCD24] provably computes the ground state energy of the SYK Hamiltonian up to an $O(1)$ -multiplicative factor for all $k < \sqrt{n}/4$.

Our key technical idea is identifying an explicit, deterministic linear operator $\mathbf{x}$ such that a fixed quadratic form of $\mathbf{x}^{2\ell}$ exactly equals the expected trace moments of the SYK Hamiltonian for every n and k . This linear operator can be naturally viewed as a *twisted* model of bosons on the space of hyperedges of a hypergraph. The problem thus reduces to identifying the spectral edge of $\mathbf{x}$, which we show is dominated by the spectrum of a natural $\binom{n}{k}$ -dimensional matrix from the *Johnson* scheme and is straightforward to compute using known results.

To show that our bound is sharp, we construct a witness state with a large quadratic form on $\mathbf{x}$ and transform it into a certificate of a lower bound on the largest quadratic form on H_{SYK} .

1 Introduction

We study the spectral edge of the *Sachdev-Ye-Kitaev* (SYK) Hamiltonian [SY93, Kit15, MS16, Sac24] in this paper. Let $n \geq k \geq 2$ be even integers, and let $\{\gamma_i\}_{1 \leq i \leq n}$ be *Majorana fermion operators*, satisfying the anti-commutation relations $\{\gamma_i, \gamma_j\} = 2\delta_{ij} \cdot \text{Id}$. Then consider the SYK Hamiltonian given as

$$H_{\text{SYK}} := \binom{n}{k}^{-1/2} \cdot \sum_{S \in \binom{[n]}{k}} g_S \Gamma_S, \quad (1)$$

where $\Gamma_S := \mathbf{i}^{k(k-1)/2} \gamma_{i_1} \cdots \gamma_{i_k}$, $S = \{i_1 < \cdots < i_k\} \subseteq [n]$ is a subset of $[n]$ of size k , $\binom{[n]}{k} := \{S \subseteq [n] : |S| = k\}$ is the collection of all k -sets in $[n]$, and $\mathbf{g} := \{g_S\}_{S \in \binom{[n]}{k}}$ is a collection of i.i.d $\mathcal{N}(0, 1)$ Gaussians, also referred to as the *disorder variables*. The normalization $\binom{n}{k}^{-1/2}$ is chosen to ensure that $\mathbb{E}_{\mathbf{g}}[H_{\text{SYK}}^2] = \text{Id}$.¹

¹This is in contrast to the standard physics convention, as the energy is not extensive.

The SYK model [SY93, Kit15] has been extensively studied in high-energy [Kit15, MS16, PR16, Ros19] and condensed-matter [SY93, CGPS22] physics, with related models studied as far back as [FW70, BF71]. The model exhibits “maximally chaotic” behavior [Kit15, MSS16, MS16], yet remains analytically tractable in certain regimes [BINT19, BM25]. The structure and complexity of (near) ground states of the SYK Hamiltonian arise naturally when studying the model as a candidate for establishing quantum speedup in simulation. This is because while there are efficient *quantum* algorithms [HO22, BCD24], there are indications that there may not be any efficient classical algorithms to compute useful descriptions of ground states of the model. The evidence for this latter claim is based on results showing that easy-to-compute candidates such as Gaussian states [HTS21, DKKA26] have energies significantly separated from the energy of states output by the above quantum algorithms.

The Ground State Energy of the SYK Model: Identifying the ground state energy (and relatedly, the spectral edge) of the SYK model itself has been an outstanding open question. Standard matrix concentration inequalities² imply that $\mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}\|_{\text{op}} \leq O(\sqrt{n})$ [FTW19, HO22]. On the other hand, in [ACKK25], the authors prove a lower bound of $\mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}\|_{\text{op}} \geq \Omega(\sqrt{n}/k)$. Interestingly, a beautiful recent work gives an efficient quantum algorithm to compute a state with an energy matching the above lower bound up to a constant factor [BCD24].³

The natural upper bound and the lower bounds above have a multiplicative gap of $\Omega(k)$ between them. Researchers have proposed both numerical computations and heuristic arguments to argue that $\mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}\|_{\text{op}}$ should in fact be $\asymp \sqrt{n}/k$ [GGJV18, HO22] matching the lower bound above. A sequence of works by Feng, Tian, and Wei [FTW19, FTW20, FTW21] made progress towards this question by understanding the *shape* (as opposed to the spectral edge) of the spectrum of the SYK model. In particular, they prove that for $k \leq o(\sqrt{n})$, the SYK model has a Gaussian spectrum [FTW19]. Furthermore, for super-constant $k \leq o(\sqrt{n})$ [GGJV18], [HO22, Section 2.9] present fine-grained heuristic calculations that suggest that $\mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}\|_{\text{op}} \sim (1 - o_n(1)) \cdot \sqrt{2n}/k$.

We note that beyond the specific quantum motivations discussed above, the problem of identifying the spectral edge of the SYK Hamiltonian can be viewed as a natural quantum analogue of determining the optimum of a classical random CSP or the injective norm of a random tensor. This is a topic with significant recent developments and applications [Fei02, WEAM25, GKM22, HKM23] and served as an inspiration for some of the technical ideas in this work.

Our Work: In this work, we rigorously confirm the above predictions and identify the spectral edge of the SYK model. Our bounds are sharp down to the right constant for large k .

Theorem 1.1 (Spectral Edge of SYK (See Theorems 5.2 and 5.3 for full version)). *Let $n \geq k \geq 2$ be even integers such that $k^2/n < 1/16$. Consider the SYK Hamiltonian H_{SYK} as in (1). Then we have*

$$\frac{\sqrt{2n}}{k} \cdot \left(1 - O(1) \cdot \max \left\{ k^{-1/2}, \frac{k^4}{n^2} \right\}\right) \leq \mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}\|_{\text{op}} \leq \frac{\sqrt{2n}}{k} + O(1).$$

In particular, if $\omega_n(1) \leq k \leq o(\sqrt{n})$, then $\mathbb{E}_{\mathbf{g}} \lambda_{\max}(H_{\text{SYK}}), \mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}\|_{\text{op}} = (1 - o_n(1)) \cdot \sqrt{2n}/k$.

² Gaussian Lipschitz concentration immediately implies a sharp concentration of $\|H_{\text{SYK}}\|$ around its mean. See Theorem 1.5.

³ For $k = 4$ [HO22] gives a simpler quantum algorithm to synthesize a state $\rho := \rho(H_{4\text{-SYK}})$ that satisfies $\mathbb{E}_{\mathbf{g}} \text{Tr}(\rho(H_{4\text{-SYK}})H_{4\text{-SYK}}) \geq \Omega(\sqrt{n})$.

- Remark 1.2.** (1) Our results, by natural modifications to our arguments, actually imply the same bounds as above on the ground state energy $\lambda_{\max}(H_{\text{SYK}})$. Observe that the upper bound follows immediately since $\lambda_{\max}(H) \leq \|H\|_{\text{op}}$. Our lower bound argument with minor modifications actually yields that a quadratic form of H_{SYK} is large and thus applies to $\lambda_{\max}(H_{\text{SYK}})$ and not just $\|H_{\text{SYK}}\|_{\text{op}}$.⁴
- (2) While our result gives an estimate that is sharp as k grows, we note that for small constant values of k , the multiplicative factor of $(\sqrt{2}/k)$ for $\sqrt{n}$ is *not* tight. For $k = 2$, the SYK Hamiltonian is exactly solvable, and in particular [MS16, FTW19] showed that $\|H_{2\text{-SYK}}\|_{\text{op}} = (\frac{4\sqrt{2}}{3\pi} + o(1))\sqrt{n} \approx 0.6\sqrt{n}$ almost surely, while $\sqrt{2n}/2 \approx 0.707\sqrt{n}$.
- (3) For $k \geq 4$, [FTW19] show that $\mathbb{E}_{\mathbf{g}} \lambda_{\max}(H_{\text{SYK}}) \leq \sqrt{\ln(2) \cdot n} \approx 0.832\sqrt{n}$. On the other hand, for $k \geq 4$ Theorem 1.1 implies that $\mathbb{E}_{\mathbf{g}} \lambda_{\max}(H_{\text{SYK}}) \leq \mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}\|_{\text{op}} \leq \frac{\sqrt{2n}}{4} + O(1) \leq 0.354\sqrt{n} + O(1)$, and thus our result outperforms that of [FTW19] for all $k \geq 4$.
- (4) For $k = 4$, [HO22] give an *algorithmic*⁵ proof that $\lambda_{\max}(H_{4\text{-SYK}}) \leq \sqrt{1 + \sqrt{6}} \cdot \sqrt{n} \approx 1.857\sqrt{n}$ with probability $\geq 1 - o_n(1)$. Very recently, [He26] established the exact spectral edge for $k = 4$, showing that almost surely $\lim_{n \rightarrow \infty} \lambda_{\max}(H_{4\text{-SYK}})/\sqrt{n} = \kappa_{\text{SD}} \approx 0.325 < \sqrt{2}/4 \approx 0.354$. On the other hand Theorem 1.1 yields $\mathbb{E}_{\mathbf{g}} \|H_{4\text{-SYK}}\|_{\text{op}} \leq 0.354\sqrt{n} + O(1)$ for all $n \geq 2$. Our techniques work for all $2 \leq k < \sqrt{n}/4$ though, and are very different from the techniques in [He26]. Furthermore, when $\omega_n(1) \leq k \leq o(\sqrt{n})$ Theorem 1.1 obtains the correct prefactor, and our results are also applicable to the sparsified SYK Hamiltonian, which we will describe shortly.

Spectral Edge of Sparse SYK Models: We show that our results extend to sparse variants of the SYK model. Formally, let $\mathcal{H} \sim \binom{[n]}{k}$ be a random hypergraph with $m = |\mathcal{H}| \ll \binom{n}{k}$ hyperedges. The sparse SYK Hamiltonian we consider is defined by:

$$H_{\text{SYK}}^{\mathcal{H}} := \frac{1}{\sqrt{|\mathcal{H}|}} \sum_{S \in \mathcal{H}} g_S \Gamma_S.$$

Understanding such sparse variants has been of interest recently [XSS20, JZL⁺22, HSHT23, DKKA26] as they may be easier to implement in practice. In this context, we care about whether relevant properties of the dense SYK model transfer to this “sparsification.” We show that the spectral edge identified in our main result above naturally extends to the sparse variants after appropriate normalization.

In the following (and throughout this paper), for any $m \geq 2$, a random k -uniform hypergraph $\mathcal{H}$ with m hyperedges is obtained by taking m independent and uniform samples from $S \in \binom{[n]}{k}$.⁶

Theorem 1.3 (Operator Norm of the Sparse Random SYK Hamiltonian (See Theorems 5.5 and 5.6 for full version)). *Let $n \geq k \geq 2$ be even integers, and suppose $k^2/n < 1/16$. Let $\mathcal{H}$ be a random*

⁴ In particular, Lemmas 4.16 to 4.18 hold directly upon replacing the operator norm $\|H_{\text{SYK}}\|_{\text{op}}$ by $\lambda_{\max}(H_{\text{SYK}})$.

⁵ More precisely, they show the following result: Let $\mathbb{R}^{\binom{[n]}{4}} \ni \mathbf{g} \mapsto \text{SoS}_8(\mathbf{g}) \in \mathbb{R}$ be the degree-8 Sum-of-Squares relaxation to estimate $\|H_{4\text{-SYK}}\|_{\text{op}}$. $\text{SoS}_8(\cdot)$ can be computed in deterministic $\text{poly}(n)$ time given $\mathbf{g}$, and furthermore, $\text{SoS}_8(\mathbf{g}) \geq \|H_{4\text{-SYK}}\|_{\text{op}}$ *pointwise* for all $\mathbf{g} \in \mathbb{R}^{\binom{[n]}{4}}$. Finally, $\mathbb{E}_{\mathbf{g}} \text{SoS}_8(\mathbf{g}) \leq \sqrt{1 + \sqrt{6}} \cdot \sqrt{n}$.

⁶ If we write the sampled set of hyperedges $\mathcal{H} = (S_1, S_2, \dots, S_m)$ with $S_i \sim \text{Unif}(\binom{[n]}{k})$ i.i.d., then the Gaussian disorder variables corresponding to S_i and S_j for $j \neq i$ are independent even though S_i and S_j might correspond to the same hyperedge.

k -uniform hypergraph with $m \geq 2^{Ck} n \log n$ edges, where $C > 0$ is some sufficiently large absolute constant. Let

$$H_{\text{SYK}}^{\mathcal{H}} := \frac{1}{\sqrt{|\mathcal{H}|}} \sum_{S \in \mathcal{H}} g_S \Gamma_S$$

be the SYK Hamiltonian corresponding to $\mathcal{H}$. Then with probability $\geq 1 - o_n(1)$ over the randomness of $\mathcal{H}$ we have

$$\frac{\sqrt{2n}}{k} \cdot \left(1 - O(1) \cdot \max \left\{ k^{-1/4}, \frac{k^4}{n^2} \right\} \right) \leq \mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} \leq \frac{\sqrt{2n}}{k} \cdot \left(1 + O(1) \cdot \max \left\{ \frac{k}{\sqrt{n}}, e^{-\Omega(k)} \right\} \right).$$

In particular if $\omega(\log(n)) \leq k \leq o(\sqrt{n})$ then $\mathbb{E} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} = (1 - o_n(1)) \cdot \sqrt{2n}/k$.

Remark 1.4. Note that bounds of the above kind are not true for *arbitrary* k -uniform hypergraphs: Indeed, consider $\mathcal{H} := \left\{ [r] \cup S : S \in \binom{[n] \setminus [r]}{k-r} \right\} \subseteq \binom{[n]}{k}$, where we assume $\omega(\log(n)) \leq k \leq o(\sqrt{n})$, and choose $r := k - \lceil 3k/\log_2(n) \rceil$ to obtain $|\mathcal{H}| \geq 2^{\Omega(k)} n \log(n)$. But at the same time we can also show that $\mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} \geq \Omega((\sqrt{n}/k) \cdot \log(n))$. Note that this happens simply because the “effective arity” of $\mathcal{H}$ is only $k - r$, which is how the spectral norm scales too.

In fact, examining our proof more closely yields the following deterministic condition that a k -uniform hypergraph $\mathcal{H}$ needs to satisfy so that we obtain tight bounds on $\mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}}$ as in [Theorem 1.3](#): Let $\mathcal{E} := |\mathcal{H}|^{-1} \cdot ((-1)^{|S \cap T|})_{S, T \in \mathcal{H}}$ be the *sign matrix* encoding the commutation structure of $\{\Gamma_S\}_{S \in \mathcal{H}}$. Also write $u := \mathbf{1}/\sqrt{|\mathcal{H}|} \in \mathbb{C}^{\mathcal{H}}$, $\Pi_{u^\perp} := \text{Id} - uu^*$, $\delta := \|\Pi_{u^\perp} \mathcal{E} u\|_2$, and suppose $\delta \leq o(1)/n$. Further suppose $1 - \langle u, \mathcal{E} u \rangle = (2 + o(1))k^2/n$, $-O(k/n) \cdot \text{Id} \preceq \Pi_{u^\perp} \mathcal{E} \Pi_{u^\perp} \preceq O(k^2/n^2) \cdot \text{Id}$. Then we have $\mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} = (1 - o_n(1))\sqrt{2n}/k$ for $\omega(\log(n)) \leq k \leq o(\sqrt{n})$.

Subgaussian Concentration for $\|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}}$. Using standard Lipschitz concentration tools in probability theory (see [Fact 3.7](#)), we can establish subgaussian concentration for $\|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}}$:

Theorem 1.5 (Subgaussian Concentration for $\|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}}$ (see [Theorems 5.4](#) and [5.7](#) for full version)). *Let $n \geq k \geq 2$ be even integers, and suppose $k^2/n < 1/16$. Fix any real $t \geq 0$. Then for $\mathcal{H} = \binom{[n]}{k}$ (the “dense case”)*

$$\Pr_{\mathbf{g}} \left(\left| \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} - \mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} \right| \geq t \right) \leq 2 \cdot \exp(-\Omega(nt^2)).$$

If $\mathcal{H}$ is a random k -uniform hypergraph with $m \geq 2^{Ck} n \log n$ edges, where $C > 0$ is some sufficiently large absolute constant, then with probability $\geq 1 - o_n(1)$ over the randomness of $\mathcal{H}$ we have

$$\Pr_{\mathbf{g}} \left(\left| \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} - \mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} \right| \geq t \right) \leq 2 \cdot \exp(-\Omega(kt^2)).$$

Remark 1.6. In the random hypergraph case, by increasing the number of hyperedges in $\mathcal{H}$ from $2^{\Omega(k)} n \log(n)$ to $2^{\Omega(k)} n^3 \log(n)$ we can get the tail decay to be $2 \cdot \exp(-\Omega(nt^2))$ as in the dense case.

Optimal Quantum Algorithms to sample SYK Ground States. As a corollary of [Theorems 1.1](#) and [1.3](#), we prove that the dissipative quantum algorithm of [\[BCD24\]](#) provably achieves an $\Omega(1)$ approximation to the ground state energy of the sparse SYK model whenever $\omega_n(1) \leq k \leq o(\sqrt{n})$. We note that for the case of $k = 4$, a recent work [\[DKKA26\]](#) gave an efficient quantum algorithm to synthesize a state $\rho(H_{4\text{-SYK}}^{\mathcal{H}})$ which achieves $\mathbb{E}_{\mathbf{g}} \text{Tr}(\rho(H_{4\text{-SYK}}^{\mathcal{H}}) H_{4\text{-SYK}}^{\mathcal{H}}) \geq \Omega(\sqrt{n})$ w.h.p. over random 4-uniform hypergraphs $\mathcal{H}$ with $m \geq \Omega(n^3 \log(n))$ hyperedges. Their

algorithm was a suitable modification of a quantum algorithm due to [HO22] who showed a similar result for the dense case $\mathcal{H} = \binom{[n]}{4}$. See also [Footnote 3](#).

Theorem 1.7 (Optimal Quantum Algorithms for the SYK Model). *Let $n \geq k \geq 2$ be even integers such that $k^2/n < 1/16$. Consider the $H_{\text{SYK}}^{\mathcal{H}}$ Hamiltonian, where we take $\mathcal{H} = \binom{[n]}{k}$ (which yields the SYK Hamiltonian (1)). Then there exists a dissipative quantum algorithm with runtime $\leq n^{O(k)}$, which on inputting $H_{\text{SYK}}^{\mathcal{H}}$ synthesizes the state $\rho(H_{\text{SYK}}^{\mathcal{H}})$ such that*

$$\mathbb{E}_{\mathbf{g}} [\text{Tr}(\rho(H_{\text{SYK}}^{\mathcal{H}}) \cdot H_{\text{SYK}}^{\mathcal{H}})] \geq \Omega(1) \cdot \mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}}. \quad (2)$$

Note that $\mathbb{E}_{\mathbf{g}} [\text{Tr}(\rho(H_{\text{SYK}}^{\mathcal{H}}) \cdot H_{\text{SYK}}^{\mathcal{H}})] \leq \mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}}$, and thus the result is optimal up to constant factors.

[Theorem 1.7](#) follows simply by combining [BCD24, Theorem 3.1] with [Theorem 1.1](#).

2 Technical Overview

We now give an exposition of the main ideas that go into the proof of our main results. Let us first focus on the upper bound for $\mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}}$, where we take $\mathcal{H}$ to be either $\binom{[n]}{k}$ or a random k -uniform hypergraph with $2^{\Omega(k)} n \log(n)$ edges.

The Trace Moment Method. A standard way to prove sharp upper bounds on the operator norms of random matrices is the trace moment method based on the following simple observation: for any integer $\ell \geq 1$ we have

$$\begin{aligned} \mathbb{E}_{\mathbf{g}} [\|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}}]^{2\ell} &\stackrel{\text{Jensen}}{\leq} \mathbb{E}_{\mathbf{g}} [\|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}}^{2\ell}] \leq \mathbb{E}_{\mathbf{g}} [\text{Tr}((H_{\text{SYK}}^{\mathcal{H}})^{2\ell})] = N \cdot \mathbb{E}_{\mathbf{g}} [\text{tr}((H_{\text{SYK}}^{\mathcal{H}})^{2\ell})] \\ &\implies \mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} \leq N^{1/(2\ell)} \cdot \mathbb{E}_{\mathbf{g}} [\text{tr}((H_{\text{SYK}}^{\mathcal{H}})^{2\ell})]^{1/(2\ell)}, \end{aligned}$$

where H_{SYK} is treated as a N -dimensional operator,⁷ and Tr, tr refer to unnormalized and normalized trace operators respectively. The advantage of this estimate is that $\mathbb{E}_{\mathbf{g}} [\text{tr}(H^{2\ell})]$ admits a combinatorial interpretation. Indeed, writing $m = |\mathcal{H}|$ we obtain that

$$\begin{aligned} \mathbb{E}_{\mathbf{g}} [\text{tr}((H_{\text{SYK}}^{\mathcal{H}})^{2\ell})] &= m^{-\ell} \cdot \sum_{S_1, \dots, S_{2\ell} \in \mathcal{H}} \mathbb{E}_{\mathbf{g}} \text{tr}(g_{S_1} \Gamma_{S_1} \cdot g_{S_2} \Gamma_{S_2} \cdots g_{S_{2\ell}} \Gamma_{S_{2\ell}}) \\ &= m^{-\ell} \cdot \sum_{S_1, \dots, S_{2\ell} \in \mathcal{H}} \mathbb{E}_{\mathbf{g}} [g_{S_1} g_{S_2} \cdots g_{S_{2\ell}}] \cdot \text{tr}(\Gamma_{S_1} \Gamma_{S_2} \cdots \Gamma_{S_{2\ell}}). \end{aligned}$$

Evaluating Gaussian moments: $\mathbb{E}_{\mathbf{g}} [g_{S_1} g_{S_2} \cdots g_{S_{2\ell}}]$ can be evaluated using the classical Isserlis-Wick theorem (see [Fact 3.4](#)) in probability theory. To invoke [Fact 3.4](#) we recall some standard terminology: Let π be a perfect matching on $[2\ell]$, i.e. $\pi = \{e_1, \dots, e_{\ell}\}$ where $\bigsqcup_{i=1}^{\ell} e_i = [2\ell]$, $|e_i| = 2$, and $e_i \cap e_j = \emptyset$ for all $1 \leq i < j \leq \ell$. The edges $e_1, \dots, e_{\ell}$ are also popularly known as *chords*, and for a chord $e = \{i < j\}$, we say i is its *left endpoint*, while j is its *right endpoint*. If we draw the numbers $1, 2, \dots, 2\ell$ on a line, and connect i, j with a chord if

⁷ It is well known that H_{SYK} can be represented as a finite dimensional operator in $N = 2^{n/2}$ -dimensional Hilbert space [BK02].

$\{i, j\} \in \pi$, then we get the *chord diagram* G_π corresponding to π . More formally, G_π is a graph whose vertices are the chords of π , and two chords $e, e' \in \pi$ are connected if they *cross* each other, i.e. if $e = \{i_1 < i_2\}$ and $e' = \{i_3 < i_4\}$ satisfy either $i_1 < i_3 < i_2 < i_4$ or $i_3 < i_1 < i_4 < i_2$.

Now [Fact 3.4](#) gives us that

$$\mathbb{E}_{\mathbf{g}}[g_{S_1} g_{S_2} \cdots g_{S_{2\ell}}] = \sum_{\pi} \prod_{e=\{i,j\} \in \pi} \mathbb{E}_{\mathbf{g}}[g_{S_i} g_{S_j}].$$

Since $\{g_S\}_{S \in \binom{[n]}{k}}$ are independent $\mathcal{N}(0, 1)$ Gaussians, we have $\mathbb{E}_{\mathbf{g}}[g_{S_i} g_{S_j}] = \mathbf{1}(S_i = S_j)$. Thus

$$\mathbb{E}_{\mathbf{g}}[\text{tr}((H_{\text{SYK}}^{\mathcal{H}})^{2\ell})] = m^{-\ell} \cdot \sum_{\pi} \sum_{S_1, \dots, S_{2\ell} \in \mathcal{H}} \mathbf{1}_{\pi}(S_1, \dots, S_{2\ell}) \cdot \text{tr}(\Gamma_{S_1} \Gamma_{S_2} \cdots \Gamma_{S_{2\ell}}),$$

where $\mathbf{1}_{\pi}(S_1, \dots, S_{2\ell}) := \prod_{e=\{i,j\} \in \pi} \mathbf{1}(S_i = S_j)$. Notice that the only tuples $(S_1, \dots, S_{2\ell})$ that contribute a non-zero term are ones where every $S \in \mathcal{H}$ occurs an even number of times.

Using Anti-Commutation Relations: Now, it follows from the anti-commutation properties of Majoranas that $\Gamma_S^2 = \text{Id}$, and $\Gamma_S \Gamma_T = \varepsilon_{S,T} \Gamma_T \Gamma_S$, where $\varepsilon_{S,T} := (-1)^{|S| \cdot |T| - |S \cap T|} = (-1)^{|S \cap T|}$. Consequently, we can take the expression $\Gamma_{S_1} \Gamma_{S_2} \cdots \Gamma_{S_{2\ell}}$ and keep swapping the matrices Γ_* s until they meet their pair under π . Consequently, $\Gamma_{S_1} \cdots \Gamma_{S_{2\ell}} = \varepsilon_{\pi} \text{Id}$, where $\varepsilon_{\pi} \in \{\pm 1\}$ is a sign which depends on π . In fact, suppose $(S_1, \dots, S_{2\ell})$ is such that $\mathbf{1}_{\pi}(S_1, \dots, S_{2\ell}) = 1$. Then for every $e = \{i, j\} \in \pi$, we have $S_i = S_j$, and thus define $S_e := S_i = S_j$. Also write $\varepsilon_{e,e'} := (-1)^{|S_e \cap S_{e'}|}$. Now note that $\varepsilon_{\pi} = \prod_{\{e,e'\} \in E(G_{\pi})} \varepsilon_{e,e'}$, and thus we have

$$\begin{aligned} \mathbb{E}_{\mathbf{g}}[\text{tr}((H_{\text{SYK}}^{\mathcal{H}})^{2\ell})] &= m^{-\ell} \cdot \sum_{\pi} \sum_{S_1, \dots, S_{2\ell} \in \mathcal{H}} \mathbf{1}_{\pi}(S_1, \dots, S_{2\ell}) \cdot \prod_{\{e,e'\} \in E(G_{\pi})} \varepsilon_{e,e'} \\ &= m^{-\ell} \cdot \sum_{\pi} \sum_{\{S_e\}_{e \in \pi}} \prod_{\{e,e'\} \in E(G_{\pi})} \varepsilon_{e,e'}, \end{aligned}$$

where we only sample ℓ elements S_e from $\mathcal{H}$, and repeat them as π dictates. But we can also absorb the $m^{-\ell}$ term into the summation to make it an expectation, and we obtain

$$\mathbb{E}_{\mathbf{g}}[\text{tr}((H_{\text{SYK}}^{\mathcal{H}})^{2\ell})] = \sum_{\pi} \mathbb{E}_{\{S_e\}_{e \in \pi} \sim \mathcal{H}} \prod_{\{e,e'\} \in E(G_{\pi})} \varepsilon_{e,e'}. \quad (3)$$

2.1 Our Key Idea: A Deterministic Moment-Matching Operator?

To illustrate our key idea, let us go back to the trace moment expansion from (3). Each term there is indexed by a pairing π , and its value is determined by the crossing graph G_{π} .

Let us focus on the dense case $\mathcal{H} = \binom{[n]}{k}$ and on a pairing whose crossing graph is a cycle to see the main insight: we can hope to relate the *expected* trace moments of H_{SYK} to the trace moments of a $\binom{n}{k}$ deterministic dimensional matrix from the *Johnson* scheme, which, in particular, allows us to read off its trace moments easily.

Write $m = |\mathcal{H}|$, and define the deterministic matrix

$$\mathcal{E} \in \mathbb{R}^{\binom{[n]}{k} \times \binom{[n]}{k}}, \quad \mathcal{E}(S, T) := \frac{1}{m} (-1)^{|S \cap T|}.$$

Suppose that G_{π} is a cycle on r chords, listed in cyclic order as $e_1, \dots, e_r$, and set $e_{r+1} := e_1$. The contribution of π to the trace-moment expansion is then

$$\mathbb{E}_{\{S_e\}_{e \in \pi} \sim \mathcal{H}} \prod_{\{e,e'\} \in E(G_{\pi})} \varepsilon_{e,e'} = \frac{1}{m^r} \sum_{S_1, \dots, S_r \in \binom{[n]}{k}} \prod_{i=1}^r (-1)^{|S_i \cap S_{i+1}|}$$

$$= \sum_{S_1, \dots, S_r \in \binom{[n]}{k}} \mathcal{E}(S_1, S_2) \cdots \mathcal{E}(S_r, S_1) = \text{Tr}(\mathcal{E}^r),$$

where $S_{r+1} := S_1$. In the expansion of the 2ℓ -th moment above, π has ℓ chords, so this identity gives $\text{Tr}(\mathcal{E}^\ell)$. Notice that the equality also includes choices for which two or more chords receive the same hyperedge, since the hyperedges are sampled independently with replacement.

If $\lambda_1, \dots, \lambda_m$ are the eigenvalues of $\mathcal{E}$, counted with multiplicity, then

$$\text{Tr}(\mathcal{E}^r) = \sum_{j=1}^m \lambda_j^r.$$

Thus, the contribution from all π where G_π is a cycle can be bounded directly in terms of the eigenvalues of $\mathcal{E}$. These eigenvalues are explicit: the entry $\mathcal{E}(S, T)$ depends only on $|S \cap T|$, and

$$\mathcal{E} = \frac{1}{m} \sum_{t=0}^k (-2)^t M_t, \quad M_t(S, T) := \binom{|S \cap T|}{t}.$$

Thus $\mathcal{E}$ belongs to the *Johnson association scheme* [God10], whose eigenspaces and eigenvalues are well understood. We have therefore expressed a part of the random SYK trace-moment expansion as a trace power of a deterministic, $\binom{n}{k}$ -dimensional operator that can be analyzed explicitly.

In the next section, we show a generalization of this simple observation by constructing an explicit deterministic operator whose spectrum controls the sum of *all* (i.e., not just the case when G_π is a cycle) the terms in (3). Like above, this deterministic operator will also be closely related to a matrix from the Johnson scheme that will help us read off the trace moment bound exactly. It turns out that our construction has a natural interpretation as a twisted model of well studied operators from the bosonic algebra. We describe this construction next.

2.2 The “ q -deformed” heuristic

Before we explain our general method to estimate this sum, it is instructive to consider the following heuristic (see [HO22] for a nice exposition): Suppose we replace

$$\mathbb{E}_{\{S_e\}_{e \in \pi} \sim \mathcal{H}} \prod_{\{e, e'\} \in E(G_\pi)} \varepsilon_{e, e'}$$

with

$$\prod_{\{e, e'\} \in E(G_\pi)} \mathbb{E}_{S_e, S_{e'} \sim \mathcal{H}} \varepsilon_{e, e'}.$$

Now note that

$$q := \mathbb{E}_{S_e, S_{e'} \sim \mathcal{H}} \varepsilon_{e, e'} = \mathbb{E}_{S_e, S_{e'} \sim \mathcal{H}} (-1)^{|S_e \cap S_{e'}|} = 1 - 2\alpha + o(\alpha),$$

where $\alpha := k^2/n$ is assumed to be $o_n(1)$.⁸ Thus substituting this in, we obtain

$$\mathbb{E}_{\mathbf{g}} [\text{tr}((H_{\text{SYK}}^{\mathcal{H}})^{2\ell})] = \sum_{\pi} \mathbb{E}_{\{S_e\}_{e \in \pi} \sim \mathcal{H}} \prod_{\{e, e'\} \in E(G_\pi)} \varepsilon_{e, e'} \approx \sum_{\pi} \prod_{\{e, e'\} \in E(G_\pi)} \mathbb{E}_{S_e, S_{e'} \sim \mathcal{H}} \varepsilon_{e, e'} = \sum_{\pi} q^{e(G_\pi)}, \quad (4)$$

⁸ In our actual proof we perform a more detailed computation and can take α to be up to a small constant, say $1/16$.

where $e(G_\pi)$ is the number of edges in G_π . We first discuss a method developed to perform this combinatorial sum, and thereafter discuss connections to the physics literature. The combinatorial problem of evaluating such chord diagrams has been studied as far back as the work of Touchard [Tou52] and Riordan [Rio75]. We also note that [GPSZ26] recently proposed a “Chord-Cavity-Generator” technique to evaluate chord diagrams rigorously.

Evaluating $\sum_\pi q^{e(G_\pi)}$. Setting aside the issue that the heuristic in Eq. (4) has not been justified yet, there is still the question of evaluating the expression $\sum_\pi q^{e(G_\pi)}$ for $q = 1 - 2\alpha + o(\alpha) \approx 1$. We follow the exposition of [BM25]. Write $[n]_q := \frac{1-q^n}{1-q}$, $[n]_q! := \prod_{j=1}^n [j]_q$, $[0]_q! := 1$, and define $\mathbf{c}, \mathbf{c}^*, T := \mathbf{c} + \mathbf{c}^*$ as

$$\mathbf{c} := \begin{pmatrix} 0 & \sqrt{[1]_q} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{[2]_q} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{[3]_q} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{[4]_q} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad \mathbf{c}^* := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \sqrt{[1]_q} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{[2]_q} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{[3]_q} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{[4]_q} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

$$T := \mathbf{c} + \mathbf{c}^* = \begin{pmatrix} 0 & \sqrt{[1]_q} & 0 & 0 & 0 & \cdots \\ \sqrt{[1]_q} & 0 & \sqrt{[2]_q} & 0 & 0 & \cdots \\ 0 & \sqrt{[2]_q} & 0 & \sqrt{[3]_q} & 0 & \cdots \\ 0 & 0 & \sqrt{[3]_q} & 0 & \sqrt{[4]_q} & \cdots \\ 0 & 0 & 0 & \sqrt{[4]_q} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Here, we imagine the rows and columns of $\mathbf{c}$ and $\mathbf{c}^*$ as indexed by $\mathbb{Z}_{\geq 0}$. Formally, $\mathbf{c}, \mathbf{c}^*$ are linear operators on $\ell_2(\mathbb{Z}_{\geq 0}) = \{(x_0, x_1, \dots, x_n, \dots) : \sum_{n=0}^\infty |x_n|^2 < \infty\}$. We write $\mathbf{f}_0 = (1 \ 0 \ 0 \ \cdots)^\top \in \ell_2(\mathbb{Z}_{\geq 0})$. Then a simple combinatorial argument⁹ [BM25] shows that $\langle \mathbf{f}_0, T^{2\ell} \mathbf{f}_0 \rangle = \sum_\pi q^{e(G_\pi)}$.

Thus, heuristically, understanding the spectral norm of T suffices to provide an upper bound on $\sum_\pi q^{e(G_\pi)}$, and thus also on $\mathbb{E}_g [\text{tr}((H_{\text{SYK}}^\mathcal{H})^{2\ell})]$. Indeed, $\mathbb{E}_g [\text{tr}((H_{\text{SYK}}^\mathcal{H})^{2\ell})]^{1/(2\ell)} \stackrel{\text{Eq. (4)}}{\leq} \langle \mathbf{f}_0, T^{2\ell} \mathbf{f}_0 \rangle^{1/(2\ell)} \leq \|T\|_{\text{op}} \stackrel{(*)}{=} 2/\sqrt{1-q} \approx \sqrt{2/\alpha} = \sqrt{2n}/k$, where $(*)$ was also shown in [BM25]. Note that $\sqrt{2n}/k$ was precisely the upper bound we were aiming for!

2.3 A Deterministic “Twisted-Bosonic” Moment-Matching Model

The key idea of our proof is to introduce a natural, tractable, *deterministic* operator $\times$ such that a natural functional of $\times$ has the *same* moments as the expected trace moments of H . We will define operators $\mathbf{a}$ and $\times := \mathbf{a} + \mathbf{a}^*$ such that $\mathbb{E}_g [\text{tr}((H_{\text{SYK}}^\mathcal{H})^{2\ell})] = \langle \mathbf{f}_0, \times^{2\ell} \mathbf{f}_0 \rangle$ for $\mathbf{f}_0$ being the vacuum state. This equality holds for all ℓ simultaneously. We can then analyze $\times$ to understand $\mathbb{E}_g [\text{tr}((H_{\text{SYK}}^\mathcal{H})^{2\ell})]$.

⁹ Note that since the only non-zero entries in T are in the first off-diagonal entries, expanding $\langle \mathbf{f}_0, T^{2\ell} \mathbf{f}_0 \rangle$ yields a sum parametrized by tuples $\tau := (n_0, \dots, n_{2\ell}) \in \mathbb{Z}_{\geq 0}^{2\ell+1}$ such that $n_0 = n_{2\ell} = 0$ and $|n_i - n_{i+1}| = 1$ for all $0 \leq i < 2\ell$. Now for any perfect matching π on $[2\ell]$, define a tuple $\tau := (n_0, \dots, n_{2\ell})$ where $n_i - n_{i-1} := 1$ if i is the left endpoint of a chord in π , and -1 otherwise. It can be shown that the term in the sum corresponding to τ equals $\sum_\pi q^{e(G_\pi)}$ over all π which produce τ as their tuple, and we are done.

Review of Bosonic Algebra: The algebra for a single boson is defined on the Fock space $\ell_2(\mathbb{Z}_{\geq 0})$ equipped with the orthonormal basis $\{f_r\}_{r \in \mathbb{Z}_{\geq 0}}$, with the annihilation and creation operators defined as,

$$b f_r = \sqrt{r} f_{r-1} \quad b^* f_r = \sqrt{r+1} f_{r+1}$$

Crucially, a single bosonic mode satisfies the commutation relation (known as *bosonic statistics*) $[b, b^*] = \text{Id}$.

In contrast, the operator c, c^* used in the evaluation of the heuristic satisfy the q -deformed commutation relation $cc^* - qc^*c = \text{Id}$.

A collection of bosons, indexed by the set of hyperedges, $\{b_S\}_{S \in \mathcal{H}}$ satisfy the mutual commutation relations (see [Definition 4.1](#) for a formal construction and [Lemma 4.2](#) for proof of commutation relations),

$$[b_S, b_T] = [b_S^*, b_T^*] = 0, \quad [b_S, b_T^*] = \mathbf{1}[S = T] \text{Id}.$$

In particular, for $S \neq T$, the respective bosonic operators commute.

Constructing a : We begin by defining a collection of operators $\{a_S\}_{S \in \mathcal{H}}$ which satisfy the same (anti)-commutation relations as $\{\Gamma_S\}_{S \in \mathcal{H}}$, i.e. $a_S a_T = \varepsilon_{S,T} a_T a_S$ and $a_S a_T^* - \varepsilon_{S,T} a_T^* a_S = \mathbf{1}[S = T] \text{Id}$ (see [Definition 4.3](#) for formal construction), where $\varepsilon_{S,T} := (-1)^{|S \cap T|}$. Note that $\varepsilon_{S,S} = 1$, thus each a_S is individually bosonic, satisfying $[a_S, a_S^*] = \text{Id}$. Their mutual statistics are determined by the signs $\varepsilon_{S,T}$, hence the *twist*. We then define $a := |\mathcal{H}|^{-1/2} \cdot \sum_{S \in \mathcal{H}} a_S$ and further $x := a + a^*$ (see [Table 1](#) for a summary of the twisted boson construction), and the key technical step in our proof (see [Proposition 4.7](#)) is the fact that

$$\langle f_0, x^{2\ell} f_0 \rangle = \mathbb{E}_g \left[\text{tr} \left((H_{\text{SYK}}^{\mathcal{H}})^{2\ell} \right) \right]. \quad (5)$$

While we shall not explain the proof of [Eq. \(5\)](#), the fact itself should not be that surprising, since the operators $\{a_S\}_{S \in \mathcal{H}}$ have been constructed to encode the same exchange signs as the SYK constituent operators $\{\Gamma_S\}_{S \in \mathcal{H}}$.

Now we proceed to justify why a is the correct “finitary” analog of c . Towards that end, let $\mathcal{E} := |\mathcal{H}|^{-1} \cdot (\varepsilon_{S,T})_{S,T \in \mathcal{H}} \in \mathbb{C}^{\mathcal{H} \times \mathcal{H}}$ be the *sign matrix*, and write

$$u := \frac{1}{\sqrt{m}} \in \mathbb{C}^{\mathcal{H}}, \quad q := \langle u, \mathcal{E} u \rangle, \quad \Pi_{u^\perp} := \text{Id} - uu^*, \quad \delta := \|\Pi_{u^\perp} \mathcal{E} u\|_2, \\ \rho^+ := \lambda_{\max}(\Pi_{u^\perp} \mathcal{E} \Pi_{u^\perp}), \quad \rho^- := \max\{0, -\lambda_{\min}(\Pi_{u^\perp} \mathcal{E} \Pi_{u^\perp})\}.$$

Looking a bit ahead, we will essentially show that $\mathcal{E} \approx quu^*$, and δ, ρ^+, ρ^- are three different ways of quantifying how much $\mathcal{E}$ actually deviates from quu^* (see [Table 2](#) for a summary of these parameters). In fact, for $k \leq O(\sqrt{n})$ we’ll show that $\rho^+, \rho^-, \delta \leq n^{-\Omega(1)}$, and thus $\mathcal{E}$ does equal quu^* as $n \rightarrow \infty$.

Bounding the Spectral Norm of x . By [Eq. \(5\)](#) it suffices to bound $\langle f_0, x^{2\ell} f_0 \rangle$. A natural way to do so is to mimic the proof in [\[BM25\]](#). Towards that, in [Lemma 4.9](#) and [Eq. \(20\)](#) we show that

$$\rho^- a^* a - (\rho^- + \delta) \cdot \sum_{S \in \mathcal{H}} a_S^* a_S \preceq aa^* - qa^* a - \text{Id} \preceq (\rho^+ + \delta) \sum_{S \in \mathcal{H}} a_S^* a_S - \rho^+ a^* a. \quad (6)$$

Now, as mentioned above, we can show that $\rho^+, \rho^-, \delta \leq n^{-\Omega(1)}$ for $k \leq O(\sqrt{n})$. The way we do this is to observe that for $\mathcal{H} = \binom{[n]}{k}$, the matrix $\mathcal{E}$ is an example of a matrix from the *Johnson*

scheme, i.e. $\mathcal{E}$ is a matrix indexed by $\binom{[n]}{k}$ such that $\mathcal{E}(S, T)$ depends only on $|S \cap T|$. The spectra of matrices in the Johnson scheme is very well-studied (see [Del73, God10, GM15, Fil16]), and thus it follows from standard computations that $\rho^+, \rho^-, \delta \leq n^{-\Omega(1)}$.¹⁰

Consequently, note that as $n \rightarrow \infty$, (6) “converges” to $aa^* - qa^*a - \text{Id} = 0$, which exactly matches the commutation relation $cc^* - qc^*c = \text{Id}$ from our heuristic that we wanted to emulate in the first place! Equivalently, (6) shows that up to an error quantified by ρ^+, ρ^-, δ , the heuristic Eq. (4) holds, and thus we obtain $\mathbb{E}_g \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} \leq \sqrt{2n/k} + \text{error}(\rho^+, \rho^-, \delta)$, thus yielding the upper bounds in Theorems 1.1 and 1.3.

Relation to Kikuchi matrices. A related random-to-Johnson comparison arises in the study of Kikuchi graphs. The Kikuchi graph of the complete uniform hypergraph belongs to the Johnson scheme, while the Kikuchi graph of a sufficiently dense random hypergraph spectrally approximates this complete-hypergraph operator [Kot26]. Kikuchi and related symmetric-difference matrices were introduced in the study of tensor PCA and random CSP refutation and have since played an important role in algorithms for random and semirandom CSPs [WEAM25, GKM22, HKM23, GHKM23]. Although our sign matrix is a different operator, the underlying principle is similar: a random-hypergraph operator can be understood by comparison with a highly symmetric operator from the Johnson scheme.

Now that we’ve proven the upper bounds in Theorems 1.1 and 1.3, we focus on proving lower bounds for $\mathbb{E}_g \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}}$. Once again, we will take inspiration from the operators c, T and Eq. (4).

2.4 Lower Bounding $\mathbb{E}_g \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}}$

Let us go back to the operators c, T , and see what they tell us about $\mathbb{E}_g \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}}$. Choose t_0 so that $q^{t_0} \leq o_n(1)$. Also choose $M \geq \omega_n(1)$. Note that for indices $t \geq t_0$, $[t]_q = (1 - q^t)/(1 - q) \approx 1/(1 - q)$. Consequently, restricting T to the indices $t_0 \leq t < t_0 + M$ yields (roughly) the following matrix:

$$\frac{1}{\sqrt{1 - q}} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

But this is simply the adjacency matrix of the path graph (a.k.a. the “hopping” model in physics), which admits the eigenvector $\left(\sin\left(\frac{(j+1)\pi}{M+1}\right)\right)_{0 \leq j < M}^\top$ and the eigenvalue $\frac{1}{\sqrt{1 - q}} \cdot 2 \cos\left(\frac{\pi}{M+1}\right) \approx \frac{2}{\sqrt{1 - q}}$, which is very close to the operator norm of T !

Once again, we are able to reproduce this construction for T to get a construction for $\mathbf{x}$. More precisely, for a specific choice of parameters t_0, M (outlined in Eq. (24) and the proof of Theorem 5.6), we show (in Lemma 4.12) that if we define $\psi_t := (a^*)^t f_0, v_t := \psi_t / \|\psi_t\|_2$, and

¹⁰ When $\mathcal{H}$ is a random hypergraph, $\mathcal{E}$ is no longer from the Johnson scheme, but standard matrix concentration results allow us to prove the necessary bounds on ρ^+, ρ^-, δ .

finally the *Krylov space*¹¹ $\mathcal{R} := \text{span}\{v_{t_0+t} : 0 \leq t < M\}$, then

$$\Pi_{\mathcal{R}} \mathbf{x} \Pi_{\mathcal{R}} = \begin{pmatrix} 0 & \beta_{t_0} & 0 & \cdots & 0 \\ \beta_{t_0} & 0 & \beta_{t_0+1} & \ddots & \vdots \\ 0 & \beta_{t_0+1} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \beta_{t_0+M-2} \\ 0 & \cdots & 0 & \beta_{t_0+M-2} & 0 \end{pmatrix},$$

where $\min_{0 \leq t < M} \beta_{t_0+t} \geq \frac{1-o(1)}{\sqrt{1-q}}$, and consequently we obtain a state (unit vector) $\psi \in \mathcal{R}$ such that $\langle \psi, \mathbf{x} \psi \rangle \geq \frac{2-o(1)}{\sqrt{1-q}} \approx \sqrt{2n}/k$.

Note that we are not yet done: ψ is only a good test function for our operator $\mathbf{x}$, but we still need to produce a certificate which achieves an energy of $\geq (1 - o(1)) \cdot \sqrt{2n}/k$ against $H_{\text{SYK}}^{\mathcal{H}}$. We use the theory of multivariate Hermite polynomials (see [Definition 3.6](#)) and Gaussian concentration (see [Lemma 4.18](#)) to map the state ψ to a certificate (see [Lemma 4.16](#)) which achieves an energy of $\approx \sqrt{2n}/k$ against $H_{\text{SYK}}^{\mathcal{H}}$, thus yielding the lower bounds in [Theorems 1.1](#) and [1.3](#), as desired.

Relation to the doubly-scaled SYK transfer matrix picture. The heuristic described in [Eq. \(4\)](#) has been studied extensively in the doubly-scaled SYK literature [[BNS18](#), [BINT19](#), [BM25](#), [GGV16](#), [GGV17](#), [GGJV18](#), [JV18](#), [Lin22](#), [LS23](#)]. Here, the doubly-scaled limit refers to,

$$n, k \rightarrow \infty, \quad \frac{k^2}{n} \rightarrow \alpha \in (0, \infty),$$

with the parameter q identified as $q = \mathbb{E}_{S,T \sim \text{Unif}(\mathcal{H})} \varepsilon_{S,T} \approx 1 - 2\alpha$. The operators c and T are then interpreted as follows: c is the annihilation operator on the “chord Hilbert space”, satisfying “deformed”¹² commutation relation $cc^* - qc^*c = \text{Id}$ with the corresponding creation operator c^* . Equivalently, the chord Hilbert-space can be defined as the span of states resulting from excitations created by c^* on the vacuum state f_0 . The transfer matrix is then exactly $T := c + c^*$, so that the RHS $\sum_{\pi} q^{e(G_{\pi})}$ evaluates to the vacuum state functional $\varphi(T^{2\ell}) := \langle f_0, T^{2\ell} f_0 \rangle$. Thus, by keeping track of the collective excitations on the chord Hilbert space, the transfer matrix picture allows moment computations in the doubly-scaled limit.

In contrast, our construction retains the full hyperedge-resolved twisted boson algebra before taking any large- n limit. Each hyperedge (interaction term in the Hamiltonian) $S \in \mathcal{H}$ is equipped with a twisted bosonic mode a_S satisfying the mutual statistics,

$$a_S a_T^* - \varepsilon_{S,T} a_T^* a_S = \mathbf{1}[S = T] \cdot \text{Id}.$$

The q -deformed oscillator c naturally emerges out of our treatment by taking the doubly scaled limit of the collective operator $a = |\mathcal{H}|^{-1/2} \sum_{S \in \mathcal{H}} a_S$. Thus, our approach can be viewed as a finite- (n, k) refinement of the transfer matrix picture in the following two aspects. First, the moment equality

$$\langle f_0, \mathbf{x}^{2\ell} f_0 \rangle = \mathbb{E}_g \left[\text{tr} \left((H_{\text{SYK}}^{\mathcal{H}})^{2\ell} \right) \right].$$

holds for any n, k and ℓ . Correlations among the individual crossing signs are retained at finite size, with the deviations from the collective mode accounted for rigorously in terms of the

¹¹ For any operator A and vector v , a space of the form $\text{span}\{A^t v : t_0 \leq t \leq t_1\}$ is known as a *Krylov space*.

¹² Note that $q = 1$ yields the usual commutation relation for bosons, while $q \neq 1$ yields the deformation.

leakage parameters $\rho^\pm$ and δ of the sign matrix $\mathcal{E}$. Second, retaining the individual modes $\{a_S\}_{S \in \mathcal{H}}$ allows for the construction of an explicit Wiener–Itô isometric embedding into the original fermionic operator algebra.¹³ This is crucial for the lower bound, as the top eigenvector for the twisted operator x is then transferred back into a disorder-dependent witness for the SYK Hamiltonian.

Prior work on “twisted” bosons. Creation and annihilation operators with twisted commutation relations belong to the general family of mixed- q Gaussian literature [BS91, FB70, BKS97, JZ15] further widely used in the doubly-scaled SYK literature [BNS18, BINT19, BM25, GGV16, GGV17, GGJV18, JV18]. In particular, [JZ15, BS94, Spe93] provide results on operators encoding a generalized commutation matrix q_{ij} . A dynamical version of the SYK model has been related to q -Brownian motion in [PS22], and convergence of the joint distribution of SYK Hamiltonians to a mixed q -Gaussian system in the large- n limit has been shown in [LS26]. Moreover, [DMM94, MMP94, MM96] investigated, in an abstract setting, operators which satisfy the twisted commutation relations our operators satisfy.

¹³ More precisely, the embedding is from the twisted Fock space into the space of Gaussian matrix-valued polynomials over the original fermionic algebra.

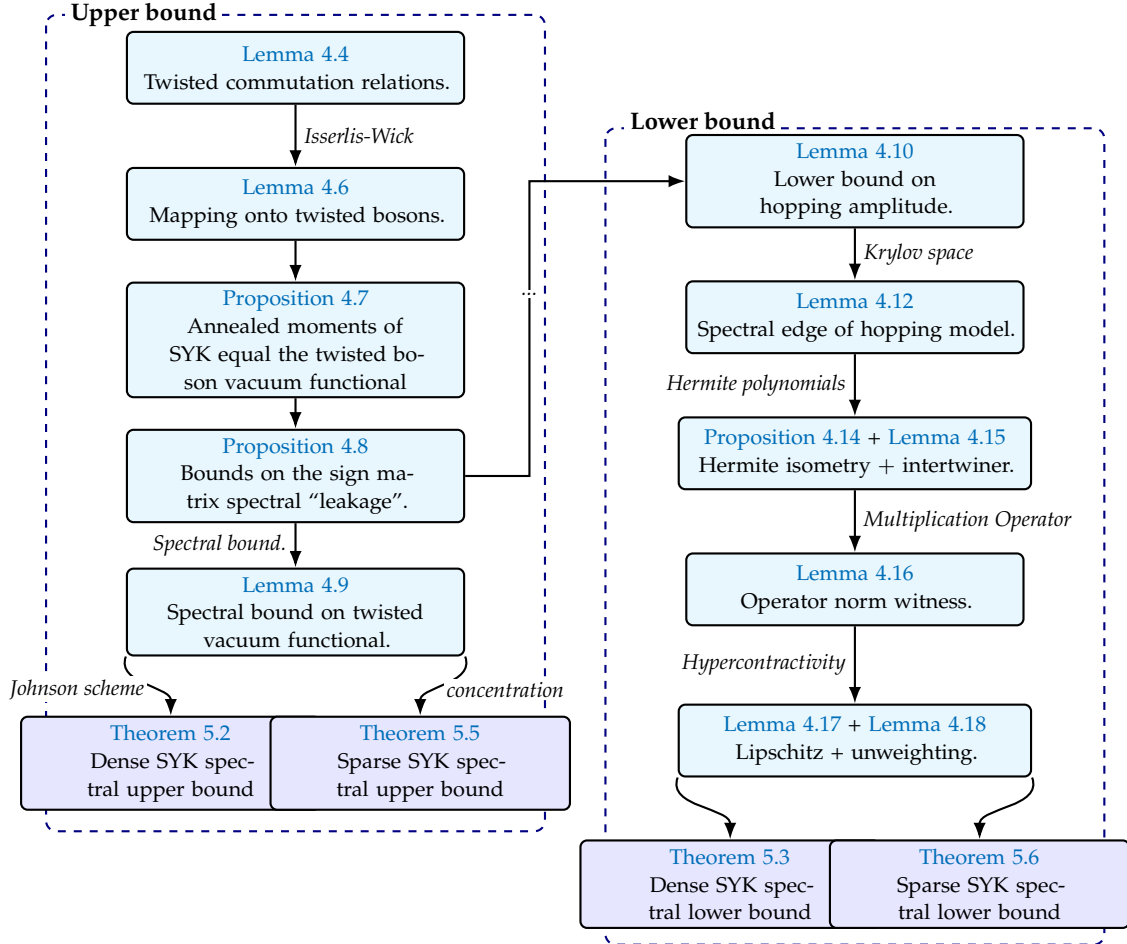


Figure 1: A map of the main results.

3 Preliminaries

Majorana Operators Let $n \geq 2$ be an even integer. Consider the *Majorana operators* $\gamma_1, \dots, \gamma_n$ satisfying the following properties:

- (1) **Hermitian** $\gamma_i = \gamma_i^*$ for all $i \in [n]$, i.e. γ_i is Hermitian,
- (2) **Involution** $\gamma_i^2 = \text{Id}$ for all $i \in [n]$,
- (3) **Anti-Commutation** $\gamma_i \gamma_j = -\gamma_j \gamma_i$ for all $i \neq j \in [n]$.

The algebra generated by $\gamma_1, \dots, \gamma_n$ admits an irreducible unitary representation into $\mathbb{C}^{N \times N}$, $N := 2^{n/2}$ [BK02].¹⁴

For any non-empty $S = \{s_1 < \dots < s_r\} \subseteq [n]$, define the Majorana monomial as,

$$\Gamma_S := \mathbf{i}^{r(r-1)/2} \gamma_{s_1} \dots \gamma_{s_r},$$

where $\mathbf{i} := \sqrt{-1}$. Let $\Gamma_\emptyset := \text{Id}$.

Using the properties of Majorana operators, we have that Γ_S is a unitary Hermitian operator with $\Gamma_S^2 = \text{Id}$. Moreover, for any two non-empty sets S, T we have

$$\Gamma_S \Gamma_T = \varepsilon_{S,T} \cdot \Gamma_T \Gamma_S. \quad (7)$$

where for any $S, T \subseteq [n]$, we define $\varepsilon_{S,T} := (-1)^{|S| \cdot |T| - |S \cap T|}$. Note that $\varepsilon_{S,T} = \varepsilon_{T,S}$ for all $S, T \subseteq [n]$, and $\varepsilon_{S,S} = 1$ for all $S \subseteq [n]$.

The SYK Hamiltonian Let $n \geq k \geq 2$ be even integers, and let $\mathcal{H}$ be a k -uniform hypergraph. Let $\{g_S\}_{S \in \mathcal{H}}$ be i.i.d $\mathcal{N}(0, 1)$ Gaussians. Then we define the *SYK Hamiltonian* to be

$$H_{\text{SYK}}^{\mathcal{H}} := \frac{1}{\sqrt{|\mathcal{H}|}} \sum_{S \in \mathcal{H}} g_S \Gamma_S. \quad (8)$$

If $\mathcal{H}$ is unspecified, $\mathcal{H}$ is to be inferred from context. The normalization ensures that $\mathbb{E}(H_{\text{SYK}}^{\mathcal{H}})^2 = \text{Id}$.

Miscellaneous Denote by $\text{tr}(\cdot)$ the normalized trace operator. For instance, $\text{tr}(\text{Id}) = 1$. Also let $\text{Tr}(\cdot)$ be usual (unnormalized) trace operator. For instance, $\text{Tr}(\text{Id}_N) = N$ for $\text{Id}_N \in \mathbb{C}^{N \times N}$. For matrices X, Y , define the commutator $[X, Y] := XY - YX$ and the anti-commutator $\{X, Y\} := XY + YX$. For a matrix X define the Hilbert-Schmidt norm as $\|X\|_{\text{HS}} := \sqrt{\text{tr}(X^* X)}$.

For a vector $v \in \mathbb{C}^N$ and a Hermitian matrix $T \in \mathbb{C}^{N \times N}$ we call $\langle v, Tv \rangle$ the (unnormalized) expectation value of T on v . The normalized expectation value defined for $v \neq 0$, is $\langle v, Tv \rangle / \langle v, v \rangle$. If T is a Hamiltonian, we call the normalized expectation value the *energy* under that Hamiltonian.

For any $\eta \in \mathbb{R}$, write $\eta_+ := \max\{\eta, 0\}$.

For any integers n, t with $n \geq 0$ define $\binom{n}{t} := 0$ if $t < 0$ or $t > n$. For $n = 0$, we define $\binom{0}{0} := 1$, and $\binom{0}{t} := 0$ for all integers $t \geq 1$. Also, for any $n \geq j \geq 1$, write $(n)_j := n(n-1) \dots (n-j+1)$. Also define $(n)_0 := 1$. Thus $\binom{n}{k} = (n)_k / k!$.

¹⁴ While there is no unique unitary irreducible representation, all irreducible representations are unitarily equivalent, and thus share the same spectrum. Since we will only be concerned with spectral properties in this paper, we fix an arbitrary such unitary irreducible representation, such as, for example, the Jordan-Wigner transform.

The Johnson Scheme Fix integers $n \geq k \geq t \geq 0$, and consider the matrix $M_t \in \mathbb{C}^{\binom{[n]}{k} \times \binom{[n]}{k}}$ given as $M_t(S, T) := \binom{|S \cap T|}{t}$. The span of the matrices $\{\sum_{t=0}^k \alpha_t M_t : \alpha_0, \dots, \alpha_k \in \mathbb{C}\}$ is the Bose-Mesner algebra of the *Johnson scheme*. For any matrix $M = \sum_{t=0}^k \alpha_t M_t$, $M(S, T)$ depends only on $|S \cap T|$.

The Johnson scheme has been intensely studied in various contexts in mathematics and computer science, and as such a detailed understanding of the spectrum of matrices in the Johnson scheme is available; see Godsil and Meagher [GM15, Chapter 6] and also Delsarte [Del73] for the foundational association-scheme treatment and Filmus [Fil16] for an explicit orthogonal basis realizing the Johnson eigenspaces. We recall the following standard fact about the spectrum of M_t for our purposes:

Fact 3.1 (Johnson eigenspaces, [Del73, Wil90]). *There exists an orthogonal decomposition $\mathbb{C}^{\binom{[n]}{k}} = \bigoplus_{j=0}^k V_j$ such that V_j is an eigenspace for M_t for all $0 \leq j, t \leq k \leq n/2$. Moreover,*

(1) $\dim(V_j) = \binom{n}{j} - \binom{n}{j-1}$ for all $j \geq 0$, where $\binom{n}{-1} := 0$. Furthermore, $V_0 = \text{span}\{\mathbf{1}\}$.

(2) $M_t|_{V_j} = \mu_{t,j} \cdot \text{Id}_{V_j}$ where

$$\mu_{t,j} := \begin{cases} \binom{k-j}{t-j} \cdot \binom{n-t-j}{k-t} & j \leq t, \\ 0 & j > t \end{cases}.$$

(3) V_j is $\mathfrak{S}_n$ -invariant for all $0 \leq j \leq k$, i.e. for any $\psi \in V_j$ and any $\sigma \in \mathfrak{S}_n$ we have $\sigma \cdot \psi \in V_j$. Here $\mathfrak{S}_n$ is the symmetric group on n points, and for any $\psi \in \mathbb{C}^{\binom{[n]}{k}}$, $\sigma \in \mathfrak{S}_n$ naturally acts on ψ as $(\sigma \cdot \psi)(S) := \psi(\sigma^{-1}(S))$, where $\sigma^{-1}(S) := \{\sigma^{-1}(i) : i \in S\}$.

The Matrix Bernstein Bound We recall the following variant of the classical matrix Bernstein bound which we will need:

Fact 3.2 (Matrix Bernstein, Theorem 7.3.1 in [Tro15]). *Let $Y_1, \dots, Y_m$ be independent mean 0 Hermitian matrices, and write $Y := \sum_{k=1}^m Y_k$. Suppose $\max_{k \in [m]} \|Y_k\|_{\text{op}} \leq R$ almost surely, and suppose $\sum_{i=1}^m \mathbb{E}[Y_i^2] \preceq \Sigma$. Write $\sigma^2 := \|\Sigma\|_{\text{op}}$, $d := 2 \text{Tr}(\Sigma)/\|\Sigma\|_{\text{op}}$. Then*

$$\Pr \left(\left\| \sum_{i=1}^m Y_i \right\|_{\text{op}} \geq t \right) \leq 2d \cdot \exp \left(-\frac{t^2}{2\sigma^2 + 2Rt/3} \right).$$

We include a somewhat specialized use-case of the above fact here: Let Ω be a set of size D . Let $\Omega' = \{\omega_1, \dots, \omega_m\}$ be a random sample of Ω of size m , where every element ω_i is chosen independently, and uniformly at random, from Ω . Note that Ω' can have repeated elements in general. Define the linear map $P : \mathbb{C}^{\Omega'} \rightarrow \mathbb{C}^{\Omega}$, where we have $P e_{\omega_i} := e_{\omega_i} \in \mathbb{C}^{\Omega}$ for all $i \in [m]$, where $\{e_{\omega_i}\}_{i \in [m]}$ (resp. $\{e_{\omega}\}_{\omega \in \Omega}$) is the standard basis for $\mathbb{C}^{\Omega'}$ (resp. $\mathbb{C}^{\Omega}$).

We remind the reader that $\text{tr}(G) = \text{Tr}(G)/D$.

Lemma 3.3. *Let $G \in \mathbb{C}^{\Omega \times \Omega}$ be a non-zero Hermitian PSD matrix with $\|G\|_{\text{op}} = \mu$. Further suppose that G has a constant diagonal, i.e. $\langle e_{\omega}, G e_{\omega} \rangle = \text{tr}(G)$ for every $\omega \in \Omega$. Then for $A_G := (D/m) P^* G P \in \mathbb{C}^{\Omega' \times \Omega'}$ we have with probability $\geq 1 - O(\exp(-100 \log(2 \text{Tr}(G)/\mu)))$ that*

$$\|A_G\|_{\text{op}} \leq O(1) \cdot \max \left\{ \mu, \frac{\text{Tr}(G)}{m} \cdot \log \left(\frac{2 \text{Tr}(G)}{\mu} \right) \right\}$$

Proof. Note that $P^* G P$ is non-zero since G has non-zero diagonal entries. Consequently,

$$\|P^* G P\|_{\text{op}} = \|G^{1/2} P P^* G^{1/2}\|_{\text{op}},$$

where we use the fact that the non-zero spectrum of BB^* and B^*B coincides for any matrix B . Furthermore, we can write $PP^* = \sum_{i=1}^m e_{\omega_i} e_{\omega_i}^*$, and thus write $X_i := (D/m)G^{1/2}e_{\omega_i}e_{\omega_i}^*G^{1/2}$, and note that

$$\|A_G\|_{\text{op}} = \frac{D}{m}\|P^*GP\|_{\text{op}} = \frac{D}{m}\|G^{1/2}PP^*G^{1/2}\|_{\text{op}} = \left\|\sum_{i=1}^m X_i\right\|_{\text{op}}.$$

Note that $\mathbb{E}[X_i] = G/m$. Also note that since X_i is rank 1, we have that $\|X_i\|_{\text{op}} = \text{Tr}(X_i) = (D/m)e_{\omega_i}^*Ge_{\omega_i} = \text{Tr}(G)/m$. Now write $Y_i := X_i - G/m$, and note that $\mathbb{E}[Y_i] = 0$. Also note that $\|Y_i\|_{\text{op}} \leq \|X_i\|_{\text{op}} + \|G\|_{\text{op}}/m \leq \text{Tr}(G)/m + \text{Tr}(G)/m = 2 \cdot \text{Tr}(G)/m$ with probability 1. Furthermore,

$$\mathbb{E}[Y_i^2] = \mathbb{E}[X_i^2] - \frac{G^2}{m^2} = \frac{\text{Tr}(G)G}{m^2} - \frac{G^2}{m^2} \preceq \frac{\text{Tr}(G)G}{m^2} \implies \sum_{i=1}^m \mathbb{E}[Y_i^2] \preceq \frac{\text{Tr}(G)G}{m}.$$

Now write

$$\sigma^2 := \left\|\frac{\text{Tr}(G)G}{m}\right\|_{\text{op}} = \frac{\mu \text{Tr}(G)}{m}, d := \frac{2 \text{Tr}\left(\frac{\text{Tr}(G)G}{m}\right)}{\left\|\frac{\text{Tr}(G)G}{m}\right\|_{\text{op}}} = \frac{2 \text{Tr}(G)}{\mu},$$

and note that by [Fact 3.2](#) we have with probability $\geq 1 - O(d^{-100})$ that

$$\begin{aligned} \|A_G\|_{\text{op}} &\leq \|G\|_{\text{op}} + \left\|\sum_{i=1}^m Y_i\right\|_{\text{op}} \\ &\leq \mu + O\left(\frac{\text{Tr}(G)}{m} \cdot \log\left(\frac{2 \text{Tr}(G)}{\mu}\right) + \sqrt{\frac{\text{Tr}(G) \cdot \mu}{m}} \cdot \sqrt{\log\left(\frac{2 \text{Tr}(G)}{\mu}\right)}\right). \end{aligned}$$

Write $\xi := \frac{\text{Tr}(G)}{m} \cdot \log\left(\frac{2 \text{Tr}(G)}{\mu}\right)$. Then we have

$$\|A_G\|_{\text{op}} \leq \mu + O(1) \cdot (\xi + \sqrt{\xi\mu}) \leq O(1) \cdot \max\{\mu, \xi\},$$

as desired. $\square$

Isserlis-Wick Theorem For any integer $\ell \geq 1$, let $\mathcal{P}_2(2\ell)$ be the set of perfect matchings on $[2\ell]$. More precisely, every $\pi \in \mathcal{P}_2(2\ell)$ is a partition of $[2\ell]$ of the form $\{\{i_1, i_2\}, \{i_3, i_4\}, \dots, \{i_{2\ell-1}, i_{2\ell}\}\}$. For any $\pi = \{\{i_1, i_2\}, \{i_3, i_4\}, \dots, \{i_{2\ell-1}, i_{2\ell}\}\} = \{e_1, e_2, \dots, e_\ell\} \in \mathcal{P}_2(2\ell)$, where $e_t := \{i_{2t-1} < i_{2t}\} \in \binom{[2\ell]}{2}$ for all $t \in [\ell]$, we define the *crossing graph* G_π of π to be a graph on the vertex set $[\ell]$, where $t_1, t_2 \in [\ell]$ form an edge in G_π if e_{t_1}, e_{t_2} cross each other. For two edges $e_1 = \{a < b\}, e_2 = \{c < d\}$, where $a, b, c, d \in [2\ell]$ are distinct elements, we say e_1, e_2 cross each other if either $a < c < b < d$ or $c < a < d < b$.

We can now recall the following classic result about moments of Gaussian random variables:

Fact 3.4 (Isserlis-Wick Theorem [[Iss18](#), [Wic50](#)]). *Let $(X_1, \dots, X_{2\ell})$ be a mean 0 Gaussian vector. Then*

$$\mathbb{E}[X_1 \cdots X_{2\ell}] = \sum_{\pi \in \mathcal{P}_2(2\ell)} \prod_{e=\{i,j\} \in \pi} \mathbb{E}[X_i X_j].$$

Probabilist's Hermite Polynomials

Definition 3.5 (Normalized Probabilist's Hermite Polynomials). For every $n \geq 0$ define the (normalized) probabilist's Hermite polynomials $h_n(X) \in \mathbb{R}[X]$ as $h_{-1}(X) := 0$, $h_0(X) := 1$, $h_1(X) := X$, where for $n \geq 1$ we have the recurrence $Xh_n(X) := \sqrt{n} \cdot h_{n-1}(X) + \sqrt{n+1} \cdot h_{n+1}(X)$.

Note that $\deg(h_n) = n$ for all $n \geq 0$. Furthermore, $\text{span}\{h_t(X) : t \leq n\} = \text{span}\{X^t : t \leq n\}$. This is also referred to as the Hermite polynomials having the *same degree filtration* as the degree monomials.

Furthermore,

$$\mathbb{E}_{g \sim \mathcal{N}(0,1)} [h_n(g)h_m(g)] = \mathbf{1}(n = m).$$

In general, we can also define multivariate Hermite polynomials:

Definition 3.6 (Normalized Multivariate Probabilist's Hermite Polynomials). For a tuple $\mathbf{r} \in \mathbb{Z}_{\geq 0}^\Lambda$, $X \in \mathbb{C}^\Lambda$, we can define $h_{\mathbf{r}}(X) := \prod_{\lambda \in \Lambda} h_{r_\lambda}(X_\lambda)$.

Then we have

$$\mathbb{E}_{g \sim \mathcal{N}(0, \text{Id}_\Lambda)} [h_{\mathbf{r}}(g)h_{\mathbf{s}}(g)] = \prod_{\lambda \in \Lambda} \mathbb{E}_{g_\lambda \sim \mathcal{N}(0,1)} [h_{r_\lambda}(g_\lambda)h_{s_\lambda}(g_\lambda)] = \mathbf{1}(\mathbf{r} = \mathbf{s}). \quad (9)$$

See [O'D21, Section 11.2] for an exposition of univariate and multivariate Hermite polynomials.

Matrix-Valued Polynomials Let $\mathcal{K}(\mathbb{R}^m; \mathbb{C}^{N \times N})$ denote all matrix-valued polynomials on $\mathbb{R}^m$. More precisely, given $\mathbf{g} = (g_1, \dots, g_m) \in \mathbb{R}^m$, a matrix-valued polynomial $p(\mathbf{g}) \in \mathcal{K}(\mathbb{R}^m; \mathbb{C}^{N \times N})$ denotes a polynomial of the form

$$\sum_{\mathbf{r} \in \mathbb{Z}_{\geq 0}^m} \left(\prod_{i=1}^m g_i^{r_i} \right) \cdot M_{\mathbf{r}},$$

where $M_{\mathbf{r}} \in \mathbb{C}^{N \times N}$, and $M_{\mathbf{r}} \neq 0$ only for finitely many $\mathbf{r} \in \mathbb{Z}_{\geq 0}^m$. For any $\mathbf{r} \in \mathbb{Z}_{\geq 0}^m$, write $|\mathbf{r}| = \sum_{i=1}^m r_i$. For any $\psi \in \mathcal{K}(\mathbb{R}^m; \mathbb{C}^{N \times N})$, we define the *degree* of ψ to be $\max_{M_{\mathbf{r}} \neq 0} |\mathbf{r}|$, where $\psi(\mathbf{g}) = \sum_{\mathbf{r} \in \mathbb{Z}_{\geq 0}^m} \left(\prod_{i=1}^m g_i^{r_i} \right) \cdot M_{\mathbf{r}}$.

We can naturally equip $\mathcal{K}(\mathbb{R}^m; \mathbb{C}^{N \times N})$ with the Gaussian measure, where we define

$$\langle F(\mathbf{g}), G(\mathbf{g}) \rangle := \mathbb{E}_{\mathbf{g} \sim \mathcal{N}(0, \text{Id}_m)} \text{tr}(F(\mathbf{g})^* G(\mathbf{g})).$$

Gaussian Concentration and Gaussian Chaos For a $\mathbb{R}$ -valued random variable X and for any $p \geq 1$, define $\|X\|_p := \mathbb{E}[|X|^p]^{1/p}$. Recall that $\|\cdot\|_p$ norms satisfy *Hölder's inequality*, i.e. if X_1, X_2 are two $\mathbb{R}$ -valued random variables, then $\|X_1 X_2\|_1 \leq \|X_1\|_p \cdot \|X_2\|_{p'}$, for any $p \geq 1$, where $p^{-1} + (p')^{-1} = 1$.

More generally, let $(V, \|\cdot\|)$ be a Hilbert space, and let X be a V -valued random variable. Then define $\|X\|_p := \mathbb{E}[\|X\|^p]^{1/p}$.

Fact 3.7 (Concentration of Lipschitz Functions [Ver18]). Let $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be a σ -Lipschitz function, i.e. $|f(x) - f(y)| \leq \sigma \cdot \|x - y\|_2$ for all $x, y \in \mathbb{R}^m$. Consider the random variable $X := f(\mathbf{g}) - \mathbb{E}_{\mathbf{g}} f(\mathbf{g})$ for $\mathbf{g} \sim \mathcal{N}(0, \text{Id}_m)$. Then we have

$$\Pr(|X| \geq t) \leq 2 \cdot \exp\left(-\Omega(1) \cdot \frac{t^2}{\sigma^2}\right).$$

In particular, for every $p \geq 2$ we have $\|X\|_p \leq O(\sigma \sqrt{p})$.

Recall that we treat $\mathbb{C}^{N \times N}$ as a Hilbert space by equipping it with the inner product $\langle X, Y \rangle := \text{tr}(X^* Y)$.

We need the following statement about Gaussian hypercontractivity. See [Jan97, O'D21] for a detailed treatment, and [Nel66, Nel73, Gro75] for a historical perspective.

Proposition 3.8 (Gaussian Hypercontractivity). *Let $Y \in \mathcal{K}(\mathbb{R}^m; \mathbb{C}^{N \times N})$ be a matrix-valued polynomial of degree $\leq L$. Define the random variable $X := Y(\mathbf{g})$, where $\mathbf{g} \sim \mathcal{N}(0, \text{Id}_m)$. Then $\|X\|_r \leq (r-1)^{L/2} \cdot \|X\|_2$ for any $r \geq 2$.*

Proof. For $\rho \in [0, 1]$, let U_ρ denote the Gaussian noise operator (see [O'D21, Definition 11.12]), defined for functions $f : \mathbb{R}^m \rightarrow \mathbb{R}$ as

$$(U_\rho f)(x) := \mathbb{E}_{\mathbf{z} \sim \mathcal{N}(0, \text{Id}_m)} f\left(\rho x + \sqrt{1 - \rho^2} \mathbf{z}\right).$$

We apply U_ρ to matrix-valued functions entrywise.

We claim that

$$\|U_\rho f\|_r \leq \|f\|_2 \tag{10}$$

for $\rho := (r-1)^{-1/2}$. Indeed, write $r' := r/(r-1)$. By duality, we have

$$\begin{aligned} \|U_\rho f\|_r &\stackrel{\text{Duality}}{=} \sup_{\|u\|_{r'} \leq 1} |\langle u, U_\rho f \rangle| \stackrel{[\text{O'D21, Section 11.1, Gaussian Hypercontractivity Theorem}]}{\leq} \sup_{\|u\|_{r'} \leq 1} |\langle u, U_\rho f \rangle| \\ &\leq \sup_{\|u\|_{r'} \leq 1} \|u\|_{1+a} \|f\|_{1+b} \end{aligned}$$

for any $a, b \geq 0$ and $0 \leq \rho \leq \sqrt{ab} \leq 1$. Choose

$$a := \frac{1}{r-1}, \quad b := 1, \quad \rho := \sqrt{ab} = \frac{1}{\sqrt{r-1}}.$$

Then we have $1+a = r'$ and $1+b = 2$. Hence

$$|\langle u, U_\rho f \rangle| \leq \|u\|_{r'} \|f\|_2 \leq \|f\|_2,$$

where the last inequality follows since while applying duality we took a supremum only over u for which $\|u\|_{r'} \leq 1$. Now, by triangle inequality,

$$\|(U_\rho F)(x)\|_{\text{HS}} = \left\| \mathbb{E}_{\mathbf{z}} F(\rho x + \sqrt{1 - \rho^2} \mathbf{z}) \right\|_{\text{HS}} \leq \mathbb{E}_{\mathbf{z}} \left\| F(\rho x + \sqrt{1 - \rho^2} \mathbf{z}) \right\|_{\text{HS}} = (U_\rho \|F(\cdot)\|_{\text{HS}})(x).$$

Applying (10) to the nonnegative scalar-valued function $x \mapsto \|F(x)\|_{\text{HS}}$ gives

$$\|U_\rho F(\mathbf{g})\|_r \leq \|F(\mathbf{g})\|_2. \tag{11}$$

Expand Y in the multivariate Hermite basis as in the definition of $\mathcal{K}(\mathbb{R}^m; \mathbb{C}^{N \times N})$ to get

$$Y(\mathbf{g}) = \sum_{\substack{\mathbf{s} \in \mathbb{Z}_{\geq 0}^m \\ |\mathbf{s}| \leq L}} h_{\mathbf{s}}(\mathbf{g}) M_{\mathbf{s}}, \quad M_{\mathbf{s}} \in \mathbb{C}^{N \times N}.$$

Such an expansion exists because the Hermite polynomials have the same degree filtration as the degree monomials. Moreover by [O'D21, Proposition 11.33] we have

$$U_\rho h_{\mathbf{s}} = \rho^{|\mathbf{s}|} h_{\mathbf{s}}.$$

Thus define

$$F(\mathbf{g}) := \sum_{\substack{\mathbf{s} \in \mathbb{Z}_{\geq 0}^m \\ |\mathbf{s}| \leq L}} \rho^{-|\mathbf{s}|} h_{\mathbf{s}}(\mathbf{g}) M_{\mathbf{s}}.$$

It follows that $U_{\rho} F = Y$. Therefore, $\|Y(\mathbf{g})\|_r \stackrel{(11)}{\leq} \|F(\mathbf{g})\|_2$.

By the orthonormality of the multivariate Hermite polynomials [O'D21, Proposition 11.33],

$$\|F(\mathbf{g})\|_{\mathcal{K},2}^2 = \mathbb{E}_{\mathbf{g}} \operatorname{tr}(F(\mathbf{g})^* F(\mathbf{g})) = \sum_{|\mathbf{s}| \leq L} \rho^{-2|\mathbf{s}|} \operatorname{tr}(M_{\mathbf{s}}^* M_{\mathbf{s}}) \leq \rho^{-2L} \sum_{|\mathbf{s}| \leq L} \operatorname{tr}(M_{\mathbf{s}}^* M_{\mathbf{s}}) = \rho^{-2L} \|Y(\mathbf{g})\|_2^2.$$

Since $\rho^{-1} = \sqrt{r-1}$, we have $\|Y(\mathbf{g})\|_r \leq \rho^{-L} \|Y(\mathbf{g})\|_2 = (r-1)^{L/2} \|Y(\mathbf{g})\|_2$, as desired. $\square$

4 The Twisted Bosonic Model

We define a bosonic model with a “twisted” phase factor. Such operators and their mathematical properties have also been studied in an abstract setting by [DMM94, MMP94, MM96].

Let us first review the standard bosonic Fock space. Given set Λ , we define for every $\mathbf{r} \in \mathbb{Z}_{\geq 0}^{\Lambda}$ the function $f_{\mathbf{r}}(\mathbf{r}') := \mathbf{1}(\mathbf{r} = \mathbf{r}')$. Write $\mathcal{F}_t := \operatorname{span}\{f_{\mathbf{r}} : |\mathbf{r}| = t\}$, and let $\mathcal{F} := \operatorname{span}\{f_{\mathbf{r}} : \mathbf{r} \in \mathbb{Z}_{\geq 0}^{\Lambda}\}$.¹⁵ We declare $\{f_{\mathbf{r}}\}_{\mathbf{r} \in \mathbb{Z}_{\geq 0}^{\Lambda}}$ to be an orthonormal basis of $\mathcal{F}$, and all subsequent adjoints are taken w.r.t. this basis.

Definition 4.1 (Bosonic Operators). For each $i \in \Lambda$, the annihilation and creation operators are defined by

$$\mathbf{b}_i f_{\mathbf{r}} := \sqrt{r_i} f_{\mathbf{r}-e_i}, \quad \mathbf{b}_i^* f_{\mathbf{r}} := \sqrt{r_i+1} f_{\mathbf{r}+e_i},$$

where, $\mathbf{r} + e_i$ (resp. $\mathbf{r} - e_i$) increments (resp. decrements, when $r_i > 0$) the i^{th} entry of $\mathbf{r}$ by 1. The number operator for mode i is defined as $N_i := \mathbf{b}_i^* \mathbf{b}_i$, and the total number operator is defined as $N := \sum_{i \in \Lambda} N_i$. Furthermore, we use the shorthand $(\pm 1)^{N_i} : \mathcal{F} \rightarrow \mathcal{F}$ to refer to the linear operator which acts as $(\pm 1)^{N_i} f_{\mathbf{r}} := (\pm 1)^{r_i} f_{\mathbf{r}}$. Note that $1^{N_i} = \operatorname{Id}$.

These satisfy the following standard commutation relations.

Lemma 4.2 (Bosonic commutation relations). *For all $i \neq j \in \Lambda$,*

$$[\mathbf{b}_i, \mathbf{b}_j] = [\mathbf{b}_i^*, \mathbf{b}_j^*] = 0, \quad [\mathbf{b}_i, \mathbf{b}_j^*] = \delta_{ij} \operatorname{Id}.$$

Moreover,

$$[\mathbf{b}_i, N_j] = \delta_{ij} \mathbf{b}_i, \quad [\mathbf{b}_i^*, N_j] = -\delta_{ij} \mathbf{b}_i^*,$$

and

$$\{\mathbf{b}_i, (-1)^{N_i}\} = \{\mathbf{b}_i^*, (-1)^{N_i}\} = 0.$$

Proof. It suffices to check the identities on $f_{\mathbf{r}}$. First,

$$\mathbf{b}_i \mathbf{b}_i^* f_{\mathbf{r}} = (r_i + 1) f_{\mathbf{r}}, \quad \mathbf{b}_i^* \mathbf{b}_i f_{\mathbf{r}} = r_i f_{\mathbf{r}},$$

¹⁵ Recall that when we take the (linear-algebraic) span of infinitely many vectors, we only look at linear combinations with finitely many non-zero coefficients, i.e. for every $f \in \mathcal{F}$ there exists a finite set T such that $f = \sum_{\mathbf{r} \in T} \alpha_{\mathbf{r}} f_{\mathbf{r}}$. Note that in physics Fock spaces are usually defined as ℓ_2 spaces in which f can have infinitely many non-zero coefficients. We choose our current definition so as to avoid functional-analytic subtleties.

Object	Definition	Interpretation
$f_{\mathbf{r}}$	$f_{\mathbf{r}}(\mathbf{r}') := \mathbf{1}(\mathbf{r} = \mathbf{r}')$	Fock basis state associated with the occupation vector $\mathbf{r} = (r_i)_{i \in \mathcal{H}}$.
b_i	$b_i f_{\mathbf{r}} := \sqrt{r_i} f_{\mathbf{r}-e_i}$	The usual bosonic annihilation operator on mode i .
N_i	$N_i := b_i^* b_i, \quad N_i f_{\mathbf{r}} = r_i f_{\mathbf{r}}$	The local number operator, which counts the number of bosons occupying mode i .
$(\pm 1)^{N_i}$	$(\pm 1)^{N_i} f_{\mathbf{r}} := (\pm 1)^{r_i} f_{\mathbf{r}}$	The occupation-parity operator on mode i .
K_i	$K_i := \prod_{j < i} \varepsilon_{ij}^{N_j}$	The twist operator associated with mode i .
a_i	$a_i := K_i b_i$	The twisted bosonic annihilation operator.

Table 1: Basic objects in the twisted boson algebra. For hyperedge $S_i \in \mathcal{H}$, and some operator O , we use the shorthand O_i for $O_{S(i)}$.

so $[b_i, b_i^*] = \text{Id}$. If $i \neq j$, then

$$b_i b_j f_{\mathbf{r}} = \sqrt{r_i r_j} f_{\mathbf{r}-e_i-e_j} = b_j b_i f_{\mathbf{r}},$$

and

$$b_i^* b_j f_{\mathbf{r}} = \sqrt{(r_i + 1) r_j} f_{\mathbf{r}+e_i-e_j} = b_j b_i^* f_{\mathbf{r}}.$$

The creation operators commute by the same calculation.

For the number operators,

$$[b_i, N_j] f_{\mathbf{r}} = \delta_{ij} \sqrt{r_i} f_{\mathbf{r}-e_i} = \delta_{ij} b_i f_{\mathbf{r}},$$

and similarly

$$[b_i^*, N_j] f_{\mathbf{r}} = -\delta_{ij} b_i^* f_{\mathbf{r}}.$$

Finally, since

$$(-1)^{N_i} f_{\mathbf{r}} = (-1)^{r_i} f_{\mathbf{r}},$$

changing r_i by one reverses its parity. Hence

$$(-1)^{N_i} b_i = -b_i (-1)^{N_i}, \quad (-1)^{N_i} b_i^* = -b_i^* (-1)^{N_i},$$

which proves the anti-commutation relations. $\square$

Now, let us define the twisted bosonic operators on the set of hyperedges $\mathcal{H}$. Let $\mathcal{H} = (S^{(1)}, \dots, S^{(m)})$ be some arbitrary ordering of $\mathcal{H}$ (where $m = |\mathcal{H}|$). Write $\varepsilon_{S^{(i)}, S^{(j)}} := \varepsilon_{ij}$.

Definition 4.3 (Twisted Bosonic Operators). For each $S^{(i)} \in \mathcal{H}$, we define the *twist operator* $K_i : \mathcal{F} \rightarrow \mathcal{F}$ as

$$K_i := \prod_{j < i} \varepsilon_{ij}^{N_j}.$$

The twisted creation and annihilation operators are then given as,

$$\mathbf{a}_i := \mathbf{K}_i \mathbf{b}_i \quad \mathbf{a}_i^* := \mathbf{K}_i \mathbf{b}_i^*$$

Individually, the twisted operators satisfy bosonic commutation relations, $[\mathbf{a}_i, \mathbf{a}_i^*] = \text{Id}$. The twisted construction is summarized in [Table 1](#).

Lemma 4.4. *The twisted bosonic operators $\{\mathbf{a}_S\}_{S \in \mathcal{H}}$ on $\mathcal{F}$ satisfy for all $S, T \in \mathcal{H}$,*

- (1) $\mathbf{a}_S \mathbf{a}_T = \varepsilon_{S,T} \mathbf{a}_T \mathbf{a}_S$,
- (2) $\mathbf{a}_S \mathbf{a}_T^* - \varepsilon_{S,T} \mathbf{a}_T^* \mathbf{a}_S = \mathbf{1}(S = T) \cdot \text{Id}$,
- (3) $\mathbf{a}_S(\mathcal{F}_t) \subseteq \mathcal{F}_{t-1}$ for all $t \geq 1$. Furthermore, $\mathbf{a}_S(\mathcal{F}_0) = 0$. Similarly, $\mathbf{a}_S^*(\mathcal{F}_t) \subseteq \mathcal{F}_{t+1}$ for all $t \geq 0$.
- (4) $\mathbf{a}_S^* \mathbf{a}_S f_{\mathbf{r}} = \mathbf{r}_S f_{\mathbf{r}}$ for all $\mathbf{r} \in \mathbb{Z}_{\geq 0}^{\mathcal{H}}$. Consequently, $\sum_{S \in \mathcal{H}} \mathbf{a}_S^* \mathbf{a}_S = \mathbf{N}$.

Proof. First, we note that $\mathbf{a}_i^2 = \mathbf{b}_i^2$ and $\varepsilon_{ii} = +1$. Secondly, note that for $1 \leq i < j \leq m$ we have,

$$\mathbf{K}_j \mathbf{b}_i = \varepsilon_{ij} \mathbf{b}_i \mathbf{K}_j$$

as $\{\mathbf{b}_i, (-1)^{\mathbf{N}_i}\} = 0$ and $[\mathbf{b}_i, (-1)^{\mathbf{N}_i}] = 0$ for $t \neq i$. Thus, for any $1 \leq i < j \leq m$ we have,

$$\mathbf{a}_i \mathbf{a}_j = \mathbf{b}_i \mathbf{K}_i \mathbf{b}_j \mathbf{K}_j = \mathbf{b}_i \mathbf{b}_j \mathbf{K}_i \mathbf{K}_j$$

and

$$\mathbf{a}_j \mathbf{a}_i = \mathbf{b}_j \mathbf{K}_j \mathbf{b}_i \mathbf{K}_i = \mathbf{b}_j (\varepsilon_{ij} \mathbf{b}_i \mathbf{K}_j) \mathbf{K}_i = \varepsilon_{ij} \mathbf{b}_i \mathbf{b}_j \mathbf{K}_i \mathbf{K}_j$$

where we used that $[\mathbf{b}_i, \mathbf{b}_j] = 0$. This shows [Item 1](#).

For [Item 2](#), first note that $\mathbf{a}_i^* \mathbf{a}_i = \mathbf{b}_i^* \mathbf{b}_i$ and $\mathbf{a}_i \mathbf{a}_i^* = \mathbf{b}_i \mathbf{b}_i^*$. For $1 \leq i < j \leq m$, the calculation proceeds exactly as [Item 1](#) since $[\mathbf{b}_i, \mathbf{b}_j^*] = 0$.

[Items 3](#) and [4](#) follow simply from the definition of the twisted bosons. $\square$

For any Hermitian $R \in \mathbb{C}^{\mathcal{H} \times \mathcal{H}}$, define

$$\mathcal{L}(R) := \sum_{S, T \in \mathcal{H}} R_{S,T} \mathbf{a}_S^* \mathbf{a}_T.$$

Also, for any two Hermitian linear operators $L, L' : \mathcal{F} \rightarrow \mathcal{F}$ we write $L \preceq L'$ if for any $t \in \mathbb{Z}_{\geq 0}$ and any $\psi \in \bigoplus_{\ell \leq t} \mathcal{F}_\ell$ we have $\langle \psi, L\psi \rangle \leq \langle \psi, L'\psi \rangle$.

We now elucidate certain properties of the operator $\mathcal{L}$:

Lemma 4.5 (Properties of $\mathcal{L}$). *We have:*

- (1) If $R \preceq R'$, then $\mathcal{L}(R) \preceq \mathcal{L}(R')$.
- (2) $\mathcal{L}(\text{Id}) = \mathbf{N}$.

Proof. Since $\mathcal{L}(R') - \mathcal{L}(R) = \mathcal{L}(R' - R)$, it suffices to show that for any $K \succcurlyeq 0$, we have $\mathcal{L}(K) \succcurlyeq 0$. Thus write $K = Q^* Q$, and note that

$$\mathcal{L}(K) = \sum_{S \in \mathcal{H}} \left(\sum_{T \in \mathcal{H}} Q_{S,T} a_T \right)^* \cdot \left(\sum_{T \in \mathcal{H}} Q_{S,T} a_T \right) \succcurlyeq 0,$$

as desired.

Note that $\mathcal{L}(\text{Id}) = \sum_{S \in \mathcal{H}} \mathbf{a}_S^* \mathbf{a}_S \stackrel{\text{Item 4 of Lemma 4.4}}{=} \mathbf{N}$. $\square$

We declare f_0 to be the *vacuum state* (since it has no particles), and we define the *vacuum functional*

$$\varphi(T) := \langle f_0, T f_0 \rangle$$

computing the expectation value of the operator T on the vacuum state $\mathbf{0} := (0, 0, \dots, 0) \in \mathbb{Z}_{\geq 0}^{\mathcal{H}}$. With this, we are ready to present the twisted bosonic mapping. The key idea is to encode the signs resulting from the anti-commutation relations of the Majorana monomials into the twist phases of the bosons.

Now, we define the collective operators

$$\mathbf{a} := \frac{1}{\sqrt{|\mathcal{H}|}} \sum_{S \in \mathcal{H}} \mathbf{a}_S, \quad \mathbf{x} := \mathbf{a} + \mathbf{a}^*.$$

Also write $\mathbf{x}_S := \mathbf{a}_S + \mathbf{a}_S^*$. Note that $\mathbf{x}$ is Hermitian. With this, we express the average moments of the SYK model in terms of vacuum expectation values of powers of $\mathbf{x}$ on the twisted bosonic space.

In [Lemma 4.6](#), the Gaussians g_{S_i} should just be thought of as a tool to introduce Isserlis-Wick-style summations.

Lemma 4.6 (Twisted Bosonic Mapping). *For any $S_1, \dots, S_{2\ell} \in \mathcal{H}$, we have*

$$\varphi(\mathbf{x}_{S_1} \cdots \mathbf{x}_{S_{2\ell}}) = \mathbb{E}[g_{S_1} g_{S_2} \cdots g_{S_{2\ell}}] \cdot \text{tr}[\Gamma_{S_1} \Gamma_{S_2} \cdots \Gamma_{S_{2\ell}}].$$

Proof. Using [Fact 3.4](#), we have

$$\mathbb{E}[g_{S_1} g_{S_2} \cdots g_{S_{2\ell}}] = \sum_{\pi \in \mathcal{P}_2(2\ell)} \prod_{\{i,j\} \in \pi} \mathbb{E}[g_{S_i} g_{S_j}] = \sum_{\pi \in \mathcal{P}_2(2\ell)} \prod_{\{i,j\} \in \pi} \mathbf{1}[S_i = S_j]$$

where we used the fact that the disorder variables are zero-mean independent Gaussians, thus $\mathbb{E}[g_{S_i} g_{S_j}] = \mathbf{1}[S_i = S_j]$.

For any perfect matching $\pi \in \mathcal{P}_2(2\ell)$ and a collection of hyperedges $S_1, \dots, S_{2\ell} \in \mathcal{H}$ define,

$$\mathbf{1}_\pi(S_1, \dots, S_{2\ell}) := \prod_{\{i,j\} \in \pi} \mathbf{1}(S_i = S_j)$$

On the event that $\mathbf{1}_\pi(S_1, \dots, S_{2\ell}) = 1$, for any edge $e \in \pi$ denote S_e to be the unique hyperedge. We define the signed crossing phase for any $\mathbf{1}_\pi(S_1, \dots, S_{2\ell}) = 1$ event to be

$$\phi_\pi(S_1, \dots, S_{2\ell}) := \prod_{\substack{\{e,f\} \subset \pi \\ e \text{ crosses } f}} \varepsilon_{e,f}$$

where the product is over unordered pairs of crossing edges, and we define $\varepsilon_{e,f} := \varepsilon_{S_e, S_f}$.

The proof will proceed in two parts. We will show,

$$\Gamma_{S_1} \cdots \Gamma_{S_{2\ell}} = \phi_\pi(S_1, \dots, S_{2\ell}) \cdot \text{Id} \text{ whenever } \mathbf{1}_\pi(S_1, \dots, S_{2\ell}) = 1 \text{ and,} \quad (\text{A})$$

$$\varphi(\mathbf{x}_{S_1} \cdots \mathbf{x}_{S_{2\ell}}) = \sum_{\pi \in \mathcal{P}_2(2\ell)} \mathbf{1}_\pi(S_1, \dots, S_{2\ell}) \cdot \phi_\pi(S_1, \dots, S_{2\ell}). \quad (\text{B})$$

Taken together, (A) and (B) imply that

$$\varphi(\mathbf{x}_{S_1} \cdots \mathbf{x}_{S_{2\ell}}) = \sum_{\pi \in \mathcal{P}_2(2\ell)} \mathbf{1}_\pi(S_1, \dots, S_{2\ell}) \text{tr}[\Gamma_{S_1} \Gamma_{S_2} \cdots \Gamma_{S_{2\ell}}]$$

proving the result.

Let us first show (A). We show this by induction on ℓ . For $\ell = 1$, this is true since $\Gamma_S^2 = \text{Id}$. Assume this is true for $\ell - 1$, and consider the case ℓ . Fix $\pi \in \mathcal{P}_2(2\ell)$ and let i be the partner of 1 in π . Then,

$$\begin{aligned} \Gamma_{S_1} \Gamma_{S_2} \cdots \Gamma_{S_i} \cdots \Gamma_{S_{2\ell}} &= \left(\prod_{j=2}^{i-1} \varepsilon_{1j} \right) \Gamma_{S_2} \cdots \underbrace{\Gamma_{S_1} \Gamma_{S_i}}_{=\text{Id}} \cdots \Gamma_{S_{2\ell}} \\ &= \left(\prod_{j=2}^{i-1} \varepsilon_{1j} \right) \cdot \phi_{\pi'}(S_2, \dots, S_{i-1}, S_{i+1}, \dots, S_{2\ell}) \cdot \text{Id} \end{aligned}$$

where $\pi' = \pi \setminus \{\{1, i\}\}$ and we used the inductive hypothesis on $\Gamma_{S_2} \cdots \Gamma_{S_{i-1}} \Gamma_{S_{i+1}} \cdots \Gamma_{S_{2\ell}}$, since $\mathbf{1}_\pi(S_1, \dots, S_{2\ell}) = 1 \implies \mathbf{1}_{\pi'}(S_2, \dots, S_{i-1}, S_{i+1}, \dots, S_{2\ell}) = 1$. Now consider a $\{p < q\} \in \pi'$ with $p < i$. If $1 < p < q < i$, then the factors satisfy $\varepsilon_{1p} \varepsilon_{1q} = 1$. Otherwise, $p < i < q$, and the edges $e = \{1 < i\}$ and $e' = \{p < q\}$ are crossing and contributes precisely $\varepsilon_{ee'}$. Therefore,

$$\left(\prod_{j=2}^{i-1} \varepsilon_{1j} \right) \cdot \phi_{\pi'}(S_2, \dots, S_{i-1}, S_{i+1}, \dots, S_{2\ell}) = \phi_\pi(S_1, S_2, \dots, S_{2\ell}).$$

This proves (A).

To show (B), first write $\mathbf{a}_i := \mathbf{a}_{S_i}$ and $\mathbf{x}_i := \mathbf{x}_{S_i} = \mathbf{a}_i + \mathbf{a}_i^*$, and consider

$$\varphi(\mathbf{x}_1 \cdots \mathbf{x}_{2\ell}) = \langle f_0, ((\mathbf{a}_1 + \mathbf{a}_1^*) \cdots (\mathbf{a}_{2\ell} + \mathbf{a}_{2\ell}^*)) f_0 \rangle$$

By [Lemma 4.4](#), we have that

$$\mathbf{a}_i \mathbf{x}_j = \mathbf{a}_i (\mathbf{a}_j + \mathbf{a}_j^*) = (\mathbf{a}_j + \mathbf{a}_j^*) \mathbf{a}_i \varepsilon_{ij} + \mathbf{1}[S_i = S_j] \text{Id} = \mathbf{x}_j \mathbf{a}_i \varepsilon_{ij} + \mathbf{1}[S_i = S_j] \text{Id} \quad (12)$$

Now, for any $f \in \mathcal{F}$, we have that $\langle f_0, (\mathbf{a} + \mathbf{a}^*) f \rangle = \langle f_0, \mathbf{a} f \rangle$ since $\mathbf{a} f_0 = 0$. We again proceed by induction. For $\ell = 1$, note that since $\mathbf{a}_i f_0 = 0$ and $\mathbf{a}_i \mathbf{a}_i^* f_0 = f_0$,

$$\langle f_0, \mathbf{x}_i \mathbf{x}_j f_0 \rangle = \langle f_0, \mathbf{a}_i \mathbf{a}_j^* f_0 \rangle = \mathbf{1}[S_i = S_j]$$

Now, assume that (B) holds for $\ell - 1$. We have, by [Eq. \(12\)](#),

$$\begin{aligned} \varphi(\mathbf{x}_1 \cdots \mathbf{x}_{2\ell}) &= \langle f_0, \mathbf{a}_1 \mathbf{x}_2 \cdots \mathbf{x}_{2\ell} f_0 \rangle \\ &= \varepsilon_{12} \langle f_0, \mathbf{x}_2 \mathbf{a}_1 \cdots \mathbf{x}_{2\ell} f_0 \rangle + \mathbf{1}[S_1 = S_2] \varphi(\widehat{\mathbf{x}}_2 \mathbf{x}_3 \cdots \mathbf{x}_{2\ell}) \end{aligned}$$

where we denote omission of an operator $\mathbf{x}$ by $\widehat{\mathbf{x}}$. Repeating this process until $\mathbf{a}_1$ is moved to the rightmost position, and subsequently invoking the inductive hypothesis

$$\begin{aligned} \varphi(\mathbf{x}_1 \cdots \mathbf{x}_{2\ell}) &= \sum_{i=2}^{2\ell} \mathbf{1}[S_1 = S_i] \left(\prod_{j<i} \varepsilon_{1j} \right) \varphi(\mathbf{x}_2 \cdots \widehat{\mathbf{x}}_i, \mathbf{x}_{i+1} \cdots \mathbf{x}_{2\ell}) \\ &= \sum_{i=2}^{2\ell} \mathbf{1}[S_1 = S_i] \left(\prod_{j<i} \varepsilon_{1j} \right) \sum_{\pi' \in \mathcal{P}_2(2\ell-2)} \mathbf{1}_{\pi'}(S_2, \dots, \widehat{S}_i, \dots, S_{2\ell}) \cdot \phi_{\pi'}(S_2, \dots, \widehat{S}_i, \dots, S_{2\ell}) \end{aligned}$$

$$= \sum_{\pi \in \mathcal{P}_2(2\ell)} \mathbf{1}_\pi(S_1, S_2, \dots, S_{2\ell}) \phi_\pi(S_1, S_2, \dots, S_{2\ell})$$

where the crossing phase factor on the partition π is determined from that of π' by the same bookkeeping as for part (A). $\square$

With [Lemma 4.6](#) we now show that the disorder-averaged moments of the SYK Hamiltonian map to the vacuum state functional of the operator $\mathbf{x}$ on the twisted bosons.

Proposition 4.7 (SYK Moments). $\varphi(\mathbf{x}^{2\ell}) = \mathbb{E} \operatorname{tr}(H^{2\ell})$, where $H = \frac{1}{\sqrt{|\mathcal{H}|}} \sum_{S \in \mathcal{H}} g_S \Gamma_S$ is the SYK Hamiltonian from [Eq. \(8\)](#).

Proof. Note that

$$\mathbb{E} \operatorname{tr}(H^{2\ell}) = \frac{1}{|\mathcal{H}|^\ell} \sum_{S_1, \dots, S_{2\ell} \in \mathcal{H}} \mathbb{E} \left[\prod_{i=1}^{2\ell} g_{S_i} \right] \cdot \operatorname{tr}(\Gamma_{S_1} \cdots \Gamma_{S_{2\ell}})$$

By [Lemma 4.6](#) we have,

$$\varphi(\mathbf{x}_{S_1} \cdots \mathbf{x}_{S_{2\ell}}) = \mathbb{E} \left[\prod_{i=1}^{2\ell} g_{S_i} \right] \cdot \operatorname{tr}(\Gamma_{S_1} \cdots \Gamma_{S_{2\ell}}).$$

Combining this with the fact that,

$$\varphi(\mathbf{x}^{2\ell}) = \frac{1}{|\mathcal{H}|^\ell} \sum_{S_1, \dots, S_{2\ell} \in \mathcal{H}} \varphi(\mathbf{x}_{S_1} \cdots \mathbf{x}_{S_{2\ell}}),$$

where $\mathbf{x}_S = \mathbf{a}_S + \mathbf{a}_S^*$, we conclude the proof. $\square$

4.1 Spectral Bounds for $\varphi(\mathbf{x}^{2\ell})$

We now provide a spectral bound for $\varphi(\mathbf{x}^{2\ell})$ (see [Lemma 4.9](#) for a formal statement). First, we do away with some necessary definitions. We write,

$$\mathcal{E} := \frac{1}{|\mathcal{H}|} (\varepsilon_{S,T})_{S,T \in \mathcal{H}} \in \mathbb{C}^{\mathcal{H} \times \mathcal{H}}, \quad (13)$$

$$u := \mathbf{1}/\sqrt{|\mathcal{H}|} = |\mathcal{H}|^{-1/2} \cdot [1 \ 1 \ \cdots \ 1]^\top, \quad (14)$$

$$\Pi_{u^\perp} := \operatorname{Id} - uu^*, \quad (15)$$

$$q := \langle u, \mathcal{E}u \rangle, \quad (16)$$

$$\delta := \|\Pi_{u^\perp} \mathcal{E}u\|, \quad (17)$$

$$\rho^+ := \lambda_{\max}(\Pi_{u^\perp} \mathcal{E} \Pi_{u^\perp}), \quad (18)$$

$$\rho^- := \max\{0, -\lambda_{\min}(\Pi_{u^\perp} \mathcal{E} \Pi_{u^\perp})\}. \quad (19)$$

We shall call $\mathcal{E}$ the *sign matrix*.

The vector u can be interpreted as the “uniform channel”, with $\rho^\pm$ measuring the leakage into the transversal channels. That is, $\delta = 0$ and $\rho^\pm = 0$ imply that $\mathcal{E} = quu^*$. The parameter δ on the other hand measures how far u is from being an eigenvector of $\mathcal{E}$; for the complete hypergraph, we have $\delta = 0$. These parameters are summarized in [Table 2](#).

Parameter	Definition	Interpretation
$\mathcal{E}$	$\mathcal{E}_{S,T} := \mathcal{H} ^{-1} \cdot \varepsilon_{S,T} = \mathcal{H} ^{-1} \cdot (-1)^{ S \cap T } \in \mathbb{C}^{\mathcal{H} \times \mathcal{H}}$	Sign matrix encoding the Majorana-monomial commutation structure.
u	$u := \mathcal{H} ^{-1/2} \cdot \mathbf{1} \in \mathbb{C}^{\mathcal{H}}$	Normalized uniform vector on the hyperedge set $\mathcal{H}$.
$\Pi_{u^\perp}$	$\Pi_{u^\perp} := \text{Id} - uu^*$	Orthogonal projection operator onto the $u^\perp$ space
δ	$\delta := \ \Pi_{u^\perp} \mathcal{E} u\ $	Quantifies the extent to which u is not an eigenvector of $\mathcal{E}$. In particular, $\delta = 0$ iff u is an eigenvector of $\mathcal{E}$.
ρ^+	$\rho^+ := \lambda_{\max}(\Pi_{u^\perp} \mathcal{E} \Pi_{u^\perp})$	The largest positive eigenvalue in the $u^\perp$ space
ρ^-	$\rho^- := \max\{0, -\lambda_{\min}(\Pi_{u^\perp} \mathcal{E} \Pi_{u^\perp})\}$	The (magnitude of the) largest negative eigenvalue in the $u^\perp$ space

Table 2: “Spectral leakage” parameters of the Sign Matrix.

Proposition 4.8. *We have*

$$(q - \delta)uu^* - (\rho^- + \delta)\Pi_{u^\perp} \preceq \mathcal{E} \preceq (q + \delta)uu^* + (\rho^+ + \delta)\Pi_{u^\perp}.$$

Proof. Consider an arbitrary $\psi \in \mathbb{C}^{\mathcal{H}}$ and write $\psi = su + y$, where $s \in \mathbb{C}$, $\langle y, u \rangle = 0$. Then

$$\begin{aligned}
\langle \psi, \mathcal{E} \psi \rangle &= |s|^2 q + 2 \operatorname{Re}(\bar{s} \cdot u^* \mathcal{E} y) + \langle y, \mathcal{E} y \rangle = |s|^2 q + 2 \operatorname{Re}(\bar{s} \cdot u^* \mathcal{E} \Pi_{u^\perp} y) + \langle y, \mathcal{E} y \rangle \\
&\leq |s|^2 q + 2 \operatorname{Re}(\bar{s} \cdot u^* \mathcal{E} \Pi_{u^\perp} y) + \rho^+ \|y\|^2 \leq |s|^2 q + 2|s| \cdot \|\Pi_{u^\perp} \mathcal{E} u\| \cdot \|y\| + \rho^+ \|y\|^2 \\
&= |s|^2 q + 2\delta |s| \cdot \|y\| + \rho^+ \|y\|^2 \leq |s|^2 q + \delta |s|^2 + \delta \|y\|^2 + \rho^+ \|y\|^2 = (q + \delta)|s|^2 + (\rho^+ + \delta)\|y\|^2 \\
&= (q + \delta)\langle \psi, uu^* \psi \rangle + (\rho^+ + \delta)\langle \psi, \Pi_{u^\perp} \psi \rangle,
\end{aligned}$$

as desired.

Similarly, we also have

$$\begin{aligned}
\langle \psi, \mathcal{E} \psi \rangle &= |s|^2 q + 2 \operatorname{Re}(\bar{s} \cdot u^* \mathcal{E} \Pi_{u^\perp} y) + \langle y, \mathcal{E} y \rangle \geq |s|^2 q - 2|s| \cdot \|\Pi_{u^\perp} \mathcal{E} u\| \cdot \|y\| - \rho^- \|y\|^2 \\
&\geq |s|^2 q - \delta |s|^2 - \delta \|y\|^2 - \rho^- \|y\|^2 \implies \mathcal{E} \succeq (q - \delta)uu^* - (\rho^- + \delta)\Pi_{u^\perp}. \quad \square
\end{aligned}$$

We now bound the vacuum functional $\varphi(x^{2\ell})$ in terms of the parameters q , ρ^+ , and δ .

Lemma 4.9 (Spectral Bounds for $\varphi(x^{2\ell})$). *If $(q - \rho^+)_+ < 1$ we have*

$$\varphi(x^{2\ell}) \leq \left(\frac{4(1 + (\rho^+ + \delta)\ell)}{1 - (q - \rho^+)_+} \right)^\ell.$$

Proof. Note that

$$\varphi(x^{2\ell}) = \langle f_0, x^{2\ell} f_0 \rangle = \|x^\ell f_0\|_2^2.$$

Let Π_ℓ be the orthogonal projection $\mathcal{F} \rightarrow \bigoplus_{0 \leq \ell' \leq \ell} \mathcal{F}_{\ell'}$. Note that by [Item 3 of Lemma 4.4](#) we have that $x^\ell f_0 = (\Pi_\ell x \Pi_\ell)^\ell f_0$. Consequently,

$$\varphi(x^{2\ell}) = \|(\Pi_\ell x \Pi_\ell)^\ell f_0\|_2^2 \leq \|\Pi_\ell x \Pi_\ell\|_{\text{op}}^{2\ell}.$$

Now, recall that $x = a + a^*$, and $a^*(\mathcal{F}_t) \subseteq \mathcal{F}_{t+1}$ for all $t \geq 0$ (by [Item 3 of Lemma 4.4](#)), to obtain that

$$\|\Pi_\ell x \Pi_\ell\|_{\text{op}} \leq 2 \max_{0 \leq t < \ell} b_t,$$

where $b_t := \|a^*\|_{\text{op}, \mathcal{F}_t \rightarrow \mathcal{F}_{t+1}}$ for $t \geq 0$ (with $b_{-1} := 0$). Thus it suffices to show

$$b_t \leq \sqrt{\frac{1 + (\rho^+ + \delta)t}{1 - (q - \rho^+)_+}}$$

for all $t \geq 0$.

Towards that end, note that

$$\begin{aligned} aa^* &= |\mathcal{H}|^{-1} \sum_{S, T \in \mathcal{H}} a_S a_T^* \stackrel{\text{Item 2 of Lemma 4.4}}{=} |\mathcal{H}|^{-1} \sum_{S, T \in \mathcal{H}} (\mathbf{1}(S = T) \text{Id} + \varepsilon_{S, T} a_T^* a_S) \\ &= \text{Id} + |\mathcal{H}|^{-1} \sum_{S, T \in \mathcal{H}} \varepsilon_{S, T} a_T^* a_S = \text{Id} + \mathcal{L}(\mathcal{E}) \\ &\stackrel{\text{Proposition 4.8 and Lemma 4.5}}{\preceq} \text{Id} + (q + \delta) \mathcal{L}(uu^*) + (\rho^+ + \delta)(\mathbf{N} - \mathcal{L}(uu^*)) \\ &= \text{Id} + (q - \rho^+) \mathcal{L}(uu^*) + (\rho^+ + \delta) \mathbf{N} = \text{Id} + (q - \rho^+) a^* a + (\rho^+ + \delta) \mathbf{N}. \end{aligned}$$

Now, note that for any $t \geq 0$, we have $(a^* a)(\mathcal{F}_t) \subseteq \mathcal{F}_t$, $(aa^*)(\mathcal{F}_t) \subseteq \mathcal{F}_t$, and $\mathbf{N}|_{\mathcal{F}_t} = t \cdot \text{Id}_{\mathcal{F}_t}$. Consequently,

$$aa^* \preceq \text{Id} + (q - \rho^+) a^* a + (\rho^+ + \delta) \mathbf{N} \implies \|aa^*\|_{\text{op}, \mathcal{F}_t} \leq 1 + (q - \rho^+)_+ \cdot \|a^* a\|_{\text{op}, \mathcal{F}_t} + (\rho^+ + \delta)t,$$

where $\|\cdot\|_{\text{op}, \mathcal{F}_t}$ refers to the operator norm restricted to $\mathcal{F}_t$. However, we also have $\|aa^*\|_{\text{op}, \mathcal{F}_t} = b_t^2$, $\|a^* a\|_{\text{op}, \mathcal{F}_t} = b_{t-1}^2$, i.e. for all $t \geq 0$ we have

$$\begin{aligned} b_t^2 &\leq (q - \rho^+)_+ \cdot b_{t-1}^2 + 1 + (\rho^+ + \delta)t \\ \implies b_t^2 &\leq \sum_{s=0}^t (q - \rho^+)_+^s \cdot (1 + (\rho^+ + \delta)(t - s)) \leq \frac{1 + (\rho^+ + \delta)t}{1 - (q - \rho^+)_+} \implies b_t \leq \sqrt{\frac{1 + (\rho^+ + \delta)t}{1 - (q - \rho^+)_+}}, \end{aligned}$$

as desired. $\square$

4.2 Towards a lower bound for the operator norm of $H_{\text{SYK}}^{\mathcal{H}}$: Spectral edge of the relevant Krylov space

In this subsection we will construct a test function which achieves energy $\approx 2/\sqrt{\nu} \approx \sqrt{2n}/k$ against x . See [Lemma 4.12](#) for a formal statement.

Note that by [Proposition 4.8](#) we know that $\mathcal{E} \succcurlyeq (q - \delta)uu^* - (\rho^- + \delta)(\Pi_{u^\perp})$. Recall from the proof of [Lemma 4.9](#) that $aa^* = \text{Id} + \mathcal{L}(\mathcal{E})$, $\mathcal{L}(uu^*) = a^* a$, and thus we have

$$aa^* \succcurlyeq \text{Id} + (q - \delta) a^* a - (\rho^- + \delta)(\mathbf{N} - a^* a) = \text{Id} + (q + \rho^-) a^* a - (\rho^- + \delta) \cdot \mathbf{N}. \quad (20)$$

Now define,

$$\psi_t := (\mathbf{a}^*)^t f_0, \quad (21)$$

$$v_t := \psi_t / \|\psi_t\|_2, \quad (22)$$

$$\beta_t := \|\psi_{t+1}\|_2 / \|\psi_t\|_2. \quad (23)$$

Note that the state f_{te_i} ¹⁶ of ψ_t will have coefficient $\sqrt{t!}/|\mathcal{H}|^t \neq 0$, so $\psi_t \neq 0$ for all $t \geq 0$. Just by definition we have $\mathbf{a}^* v_t = \beta_t v_{t+1}$, $\beta_0 = 1$. As we will show soon, β_t act as effective “hopping amplitudes” on the relevant Krylov space spanned by the v_t . Also note that $\psi_t, v_t \in \mathcal{F}_t$. The states ψ_t represent collective excitations of the twisted bosons.

Lemma 4.10 (Lower bound on hopping amplitude). *Assume $0 \leq q < 1$. Then*

$$\beta_t^2 \geq \frac{1 - q^{t+1} - (\rho^- + \delta)t}{1 - q}.$$

Proof. Taking the expectation value of $\mathbf{a}\mathbf{a}^*$ on v_t , by (20) we have

$$\langle v_t, \mathbf{a}\mathbf{a}^* v_t \rangle \geq 1 + (q + \rho^-) \langle v_t, \mathbf{a}^* \mathbf{a} v_t \rangle - (\rho^- + \delta) \cdot \langle v_t, \mathbf{N} v_t \rangle.$$

Now, $\langle v_t, \mathbf{a}\mathbf{a}^* v_t \rangle = \|\mathbf{a}^* v_t\|^2 = \beta_t^2$, since $\|v_{t+1}\|_2 = 1$. Also, $\langle v_t, \mathbf{a}^* \mathbf{a} v_t \rangle = \|\mathbf{a} v_t\|^2$. Since $v_t \in \mathcal{F}_t$ we also have $\langle v_t, \mathbf{N} v_t \rangle = t$. Consequently,

$$\beta_t^2 \geq 1 + (q + \rho^-) \|\mathbf{a} v_t\|^2 - (\rho^- + \delta)t \geq 1 + q \|\mathbf{a} v_t\|^2 - (\rho^- + \delta)t.$$

Now note that $\langle \mathbf{a} v_t, v_{t-1} \rangle = \langle v_t, \mathbf{a}^* v_{t-1} \rangle = \beta_{t-1} \langle v_t, v_{t-1} \rangle = \beta_{t-1}$. Consequently, $\mathbf{a} v_t = \beta_{t-1} v_{t-1} + z$, where $\langle z, v_{t-1} \rangle = 0$. In particular, $\|\mathbf{a} v_t\|^2 = \beta_{t-1}^2 + \|z\|^2 \geq \beta_{t-1}^2$, and consequently,

$$\begin{aligned} \beta_t^2 \geq 1 + q\beta_{t-1}^2 - (\rho^- + \delta)t &\implies \beta_t^2 \geq \sum_{s=0}^t q^s - (\rho^- + \delta) \sum_{r=1}^t r q^{t-r} \\ &\geq \frac{1 - q^{t+1}}{1 - q} - (\rho^- + \delta)t \sum_{r=0}^{t-1} q^r \geq \frac{1 - q^{t+1}}{1 - q} - \frac{(\rho^- + \delta)t}{1 - q}, \end{aligned}$$

as desired. $\square$

Define the parameters

$$\nu := 1 - q, \quad B := \sqrt{\nu/(\rho^- + \delta)}, \quad M := \lfloor 1/\nu \rfloor. \quad (24)$$

Definition 4.11 (Krylov Space). For parameters t_0, M define the subspace (on the twisted Fock space)

$$\mathcal{R}_{t_0, M} := \text{span}\{v_t : t_0 \leq t < t_0 + M\} \quad (25)$$

and $\Pi_{\mathcal{R}_{t_0, M}}$ to be the orthogonal projector onto $\mathcal{R}_{t_0, M}$. If t_0, M are understood from context we will not mention it explicitly.

The collective $\times$ operator projected onto $\mathcal{R}$ resembles a one-dimensional “hopping model” with nearly-constant hopping. The spectral edge of this hopping model scales as $2/\sqrt{\nu}$.

¹⁶ $te_i \in \mathbb{Z}_{\geq 0}^{\mathcal{H}}$ refers to the tuple with t particles on the i^{th} hyperedge.

Lemma 4.12 (Spectral edge of hopping model). *Suppose $B \geq \kappa \geq \Omega_n(1)$ for some parameter κ . Write $t_0 := \lceil \kappa/\nu \rceil$, $L := t_0 + M$. Then*

$$\lambda_{\max}(\Pi_{\mathcal{R}} \times \Pi_{\mathcal{R}}) \geq \frac{2}{\sqrt{\nu}} \cdot (1 - O(1) \cdot \max\{\kappa^{-1}, \nu^2\}).$$

Proof. Note that for $t_0 \leq t < t_0 + M$ we have $q^{t+1} \leq e^{-\nu(t+1)} \leq e^{-\kappa}$, $(\rho^- + \delta)t \leq (\rho^- + \delta)(\kappa + 2)/\nu$. Since $\kappa \geq \Omega(1)$ we have $(\rho^- + \delta)t \leq O(1) \cdot (\rho^- + \delta)\kappa/\nu \leq O(1) \cdot (\nu/\kappa^2)(\kappa/\nu) \leq O(1/\kappa)$, and thus $q^{t+1} + (\rho^- + \delta)t \leq e^{-\kappa} + O(1/\kappa) \leq O(1/\kappa)$.

Now, by the definition of $\mathbf{x}$ we have that

$$\Pi_{\mathcal{R}} \times \Pi_{\mathcal{R}} = \begin{pmatrix} 0 & \beta_{t_0} & 0 & \cdots & 0 \\ \beta_{t_0} & 0 & \beta_{t_0+1} & \ddots & \vdots \\ 0 & \beta_{t_0+1} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \beta_{t_0+M-2} \\ 0 & \cdots & 0 & \beta_{t_0+M-2} & 0 \end{pmatrix},$$

where this matrix is expressed in the orthonormal basis $\{v_t\}_{t_0 \leq t < t_0 + M}$ defining $\mathcal{R}$.

Now write

$$\psi := \sum_{t=0}^{M-1} \sin\left(\frac{(t+1)\pi}{M+1}\right) \cdot v_{t_0+t} \in \mathcal{R}.$$

Since every entry of ψ is non-negative, by [Lemma 4.10](#) we have

$$\begin{aligned} \lambda_{\max}(\Pi_{\mathcal{R}} \times \Pi_{\mathcal{R}}) &\geq \frac{\langle \psi, \Pi_{\mathcal{R}} \times \Pi_{\mathcal{R}} \psi \rangle}{\langle \psi, \psi \rangle} \geq \frac{1 - O(1/\kappa)}{\sqrt{\nu}} \cdot 2 \cos\left(\frac{\pi}{M+1}\right) \\ &\geq \frac{2}{\sqrt{\nu}} \cdot (1 - O(1) \cdot \max\{\kappa^{-1}, \nu^2\}), \end{aligned}$$

as desired.¹⁷ □

Remark 4.13. *A priori*, increasing L improves the bound on the spectral edge. However, for subsequent applications of Gaussian Hypercontractivity ([Proposition 3.8](#)) to obtain the spectral lower bound in [Section 4.3](#), L will play the role of degree of a polynomial, which we require to be $n^{O(1)}$ (see [Theorem 5.3](#)).

4.3 A lower bound for $\|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}}$: Explicit Mapping from Twisted Boson Space to Fermionic Space

We now transfer a lower bound on $\lambda_{\max}(\Pi_{\mathcal{R}} \times \Pi_{\mathcal{R}})$ to a lower bound on $\mathbb{E}_g \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}}$. See [Lemma 4.18](#) for a formal statement.

Recall [Definition 3.5](#), and recall that for any $\mathbf{r} \in \mathbb{Z}_{\geq 0}^{\mathcal{H}}$, $\mathbf{g} \in \mathbb{R}^{\mathcal{H}}$ we define $h_{\mathbf{r}}(\mathbf{g}) := \prod_{S \in \mathcal{H}} h_{\mathbf{r}_S}(\mathbf{g}_S)$. Also fix some ordering $\mathcal{H} = (S_1, \dots, S_m)$ (where $m = |\mathcal{H}|$) and define $\Gamma^{\mathbf{r}} := \Gamma_{S_1}^{\mathbf{r}_1} \cdots \Gamma_{S_m}^{\mathbf{r}_m}$.

¹⁷ Note that $2 \cos(\pi/(M+1))$ is the largest eigenvalue of the adjacency matrix of the path graph on M vertices, which is what we get once we replace all the β_* s with their lower bound. Note that we can do this comparison without worry since ψ has non-negative coefficients for all v_t .

Now define the linear map $U : \mathcal{F} \rightarrow \mathcal{K}(\mathbb{R}^{\mathcal{H}}, \mathbb{C}^{N \times N})$ given as

$$(Uf_{\mathbf{r}})(\mathbf{g}) := h_{\mathbf{r}}(\mathbf{g}) \cdot \Gamma^{\mathbf{r}}. \quad (26)$$

We first show that this map is an isometric embedding of the twisted bosons into fermionic space. This isometry, for usual bosons, is an instance of the *Wiener-Itô chaos decomposition* [Wie38, Itô51]. Incorporating the twists into this decomposition is fairly straightforward though, as shown below:

Proposition 4.14 (Isometric Embedding). *For any $\psi, \psi' \in \mathcal{F}$, $\langle \psi, \psi' \rangle_{\mathcal{F}} = \langle U\psi, U\psi' \rangle_{\mathcal{K}}$.*

Proof. Observe that

$$\langle Uf_{\mathbf{r}}, Uf_{\mathbf{s}} \rangle_{\mathcal{K}} = \mathbb{E}_{\mathbf{g}} \langle h_{\mathbf{r}}(\mathbf{g}) \Gamma^{\mathbf{r}}, h_{\mathbf{s}}(\mathbf{g}) \Gamma^{\mathbf{s}} \rangle \stackrel{\text{Eq. (9)}}{=} \mathbf{1}(\mathbf{r} = \mathbf{s}) \cdot \langle \Gamma^{\mathbf{r}}, \Gamma^{\mathbf{s}} \rangle = \mathbf{1}(\mathbf{r} = \mathbf{s}).$$

Here we use the fact that $\Gamma_S^2 = \text{Id}$ for all non-empty $S \subseteq [n]$. Since U is a linear map which maps an orthonormal basis of $\mathcal{F}$ to an orthonormal set in $\mathcal{K}$, we are done. $\square$

Also, we define the Hamiltonian $H(\mathbf{g}) \in \mathcal{K}(\mathbb{R}^{\mathcal{H}}, \mathbb{C}^{N \times N})$ with coefficients given by $\mathbf{g}$:

$$H(\mathbf{g}) := \frac{1}{\sqrt{m}} \sum_{S \in \mathcal{H}} \mathbf{g}_S \Gamma_S.$$

Finally define the left multiplication operator $\mathcal{M} : \mathcal{K}(\mathbb{R}^{\mathcal{H}}, \mathbb{C}^{N \times N}) \rightarrow \mathcal{K}(\mathbb{R}^{\mathcal{H}}, \mathbb{C}^{N \times N})$ given as

$$(\mathcal{M}F)(\mathbf{g}) := H(\mathbf{g}) \cdot F(\mathbf{g}).$$

Note that $\mathcal{M}$ is linear. We now show that the embedding U as defined in Eq. (26) satisfies an intertwining property with respect to the Hamiltonian $H(\mathbf{g})$. This intertwining property essentially follows from the fact that Hermite polynomials and creation-annihilation operators in bosonic spaces satisfy the same recurrence; Indeed, Hermite polynomials satisfy the recurrence $g_i h_{\mathbf{r}}(\mathbf{g}) = \sqrt{\mathbf{r}_i + 1} h_{\mathbf{r}+e_i}(\mathbf{g}) + \sqrt{\mathbf{r}_i} h_{\mathbf{r}-e_i}(\mathbf{g})$, while bosonic creation-annihilation operators satisfy the recurrence $(b_i + b_i^*) f_{\mathbf{r}} = \sqrt{\mathbf{r}_i + 1} f_{\mathbf{r}+e_i} + \sqrt{\mathbf{r}_i} f_{\mathbf{r}-e_i}$. This correspondence is known as the Wiener-Itô-Segal transform [Wie38, Itô51, Seg56]. Note that while the Wiener-Itô-Segal transform is stated only for usual bosons, incorporating the twists in our twisted bosonic system into the transform is fairly straightforward, as shown below. For a modern treatment of both the Wiener-Itô chaos decomposition stated earlier, and the Wiener-Itô-Segal transform, we refer the reader to the textbook of Janson [Jan97]. For a physics perspective, see [Sim74, Chapter 1].

Lemma 4.15 (Intertwining). *We have $\mathcal{M}U = U \times$ as linear maps $\mathcal{F} \rightarrow \mathcal{K}(\mathbb{R}^{\mathcal{H}}, \mathbb{C}^{N \times N})$.*

Proof. It suffices to show that $\mathcal{M}Uf_{\mathbf{r}} = U \times f_{\mathbf{r}}$ for all $\mathbf{r} \in \mathbb{Z}_{\geq 0}^{\mathcal{H}}$. Towards that end note that

$$(\mathcal{M}Uf_{\mathbf{r}})(\mathbf{g}) = \frac{1}{\sqrt{m}} \left(\sum_{i=1}^m \mathbf{g}_{S_i} \Gamma_{S_i} \right) \cdot h_{\mathbf{r}}(\mathbf{g}) \Gamma^{\mathbf{r}} = \frac{1}{\sqrt{m}} \sum_{i=1}^m \mathbf{g}_{S_i} h_{\mathbf{r}}(\mathbf{g}) \Gamma_{S_i} \Gamma^{\mathbf{r}}.$$

Using the commutation relations $\Gamma_S \Gamma_T = \varepsilon_{S,T} \Gamma_T \Gamma_S$ repeatedly we obtain $\Gamma_{S_i} \Gamma^{\mathbf{r}} = \sigma_i(\mathbf{r}) \Gamma^{\mathbf{r}+e_i}$, where $\sigma_i(\mathbf{r}) \equiv \prod_{j < i} \varepsilon_{ij}^{\mathbf{r}_j}$ is the twist phase. Furthermore, by Definition 3.5, we have $g_i h_{\mathbf{r}}(\mathbf{g}) = \sqrt{\mathbf{r}_i + 1} h_{\mathbf{r}+e_i}(\mathbf{g}) + \sqrt{\mathbf{r}_i} h_{\mathbf{r}-e_i}(\mathbf{g})$. Thus

$$(\mathcal{M}Uf_{\mathbf{r}})(\mathbf{g}) = \frac{1}{\sqrt{m}} \sum_{i=1}^m (\sqrt{\mathbf{r}_i + 1} h_{\mathbf{r}+e_i}(\mathbf{g}) + \sqrt{\mathbf{r}_i} h_{\mathbf{r}-e_i}(\mathbf{g})) \cdot \sigma_i(\mathbf{r}) \Gamma^{\mathbf{r}+e_i}.$$

Now note that since $\Gamma_S^2 = \text{Id}$, we have $\Gamma^{r+e_i} = \Gamma^{r-e_i}$ when $r_i \geq 1$. Thus

$$\begin{aligned} (\mathcal{M}Uf_r)(g) &= \frac{1}{\sqrt{m}} \sum_{i=1}^m \sigma_i(r) \cdot (\sqrt{r_i+1} \cdot (Uf_{r+e_i})(g) + \sqrt{r_i} \cdot (Uf_{r-e_i})(g)) \\ &= \frac{1}{\sqrt{m}} \sum_{i=1}^m (U(\sqrt{r_i+1}\sigma_i(r) \cdot f_{r+e_i}))(g) + U(\sqrt{r_i}\sigma_i(r) \cdot f_{r-e_i})(g) \\ &\implies \mathcal{M}Uf_r = U \cdot \frac{1}{\sqrt{m}} \sum_{i=1}^m (\sqrt{r_i+1}\sigma_i(r) \cdot f_{r+e_i} + \sqrt{r_i}\sigma_i(r) \cdot f_{r-e_i}). \end{aligned}$$

Thus it suffices to show that $\sqrt{r_i+1}\sigma_i(r) \cdot f_{r+e_i} + \sqrt{r_i}\sigma_i(r) \cdot f_{r-e_i} = (a_{S_i} + a_{S_i}^*)f_r$. But this follows from the definitions. $\square$

We now use the multiplication operator to yield a lower bound for $\|H(g)\|_{\text{op}}$ multiplied by a “tilt” $w(g)$. Subsequently in an upcoming Lemma we use Gaussian hypercontractivity to decouple the tilt $w(g)$ from $\|H(g)\|_{\text{op}}$ to yield a lower bound just for $\mathbb{E}_{g \sim \mathcal{N}(0, \text{Id}_m)} \|H(g)\|_{\text{op}}$. The technique of decoupling a product using hypercontractivity is a standard method in the analysis of boolean functions, see [O’D21] for further perspectives.

Lemma 4.16 (Operator Norm Witness). *Let $\mathcal{R}$ be any space of the kind defined in Definition 4.11. Then there exists $F \in \mathcal{K}(\mathbb{R}^{\mathcal{H}}, \mathbb{C}^{N \times N})$ of degree $\leq L$ such that*

$$\mathbb{E}_{g \sim \mathcal{N}(0, \text{Id}_m)} [\|H(g)\|_{\text{op}} \cdot w(g)] \geq \lambda_{\max}(\Pi_{\mathcal{R}} \times \Pi_{\mathcal{R}}),$$

where:

- (1) $w(g) := \|F(g)\|_{\text{HS}}^2$,
- (2) More explicitly, we can choose $F = U\psi$, where $\psi \in \mathcal{R}$ is a unit vector for which $\langle \psi, \mathbf{x}\psi \rangle_{\mathcal{F}} = \lambda_{\max}(\Pi_{\mathcal{R}} \times \Pi_{\mathcal{R}})$,
- (3) $\mathbb{E}_g w(g) = 1$.

Proof. By Proposition 4.14 we have $\langle \psi, \mathbf{x}\psi \rangle_{\mathcal{F}} = \langle U\psi, U\mathbf{x}\psi \rangle_{\mathcal{K}}$. By Lemma 4.15 we have $\langle U\psi, U\mathbf{x}\psi \rangle_{\mathcal{K}} = \langle U\psi, \mathcal{M}U\psi \rangle_{\mathcal{K}}$. Write $F := U\psi$. Then

$$\langle U\psi, \mathcal{M}U\psi \rangle_{\mathcal{K}} = \langle F, \mathcal{M}F \rangle_{\mathcal{K}} = \mathbb{E}_{g \sim \mathcal{N}(0, \text{Id}_m)} \text{tr}(F(g)^*(\mathcal{M}F)(g)) = \mathbb{E}_{g \sim \mathcal{N}(0, \text{Id}_m)} \text{tr}(F(g)^*H(g)F(g)).$$

Since $H(g) \preceq \|H(g)\|_{\text{op}} \cdot \text{Id}$, we have $F(g)^*H(g)F(g) \preceq \|H(g)\|_{\text{op}} \cdot F(g)^*F(g)$, and thus $\text{tr}(F(g)^*H(g)F(g)) \leq \|H(g)\|_{\text{op}} \cdot \text{tr}(F(g)^*F(g)) = \|H(g)\|_{\text{op}} \cdot w(g)$. Finally, note that since $\psi \in \bigoplus_{t \leq L} \mathcal{F}_t$, $F \in \mathcal{K}$ has degree $\leq L$ by the definition of U . Furthermore, $\mathbb{E}_g w(g) = \langle F, F \rangle_{\mathcal{K}} = \langle \psi, \psi \rangle_{\mathcal{F}} = 1$; thus we are done. $\square$

We now show that the Hamiltonian $H(g)$ is Lipschitz in g . This has already been shown in [ACKK25, FTW20], where further connections to concepts such as the commutator index and the Lovász theta function of the commutator graph have been drawn. For our purposes the following simple spectral argument will suffice.

Lemma 4.17 (Lipschitzness). *For every $g, g' \in \mathbb{R}^{\mathcal{H}}$ we have*

$$\|H(g) - H(g')\|_{\text{op}} \leq \sqrt{\frac{\rho^+ + \delta}{1 - q}} \cdot \|g - g'\|_2,$$

where q, ρ^+, δ are as in Eq. (13), assuming $q < 1$.

Proof. Consider an arbitrary $\mathbf{g} \in \mathbb{R}^{\mathcal{H}}$, $\mathbf{g} \neq 0$. Since $H(\mathbf{g}) - H(\mathbf{g}') = H(\mathbf{g} - \mathbf{g}')$, it suffices to show that $\|H(\mathbf{g})\|_{\text{op}} \leq \sqrt{\frac{\rho^+ + \delta}{1-q}} \cdot \|\mathbf{g}\|_2$. Let $\psi \in \mathbb{C}^N$ be an arbitrary unit vector, and define the matrix $X_\psi := X \in \mathbb{R}^{\mathcal{H} \times \mathcal{H}}$ defined as $X_{S,T} := \mathbf{g}_S \mathbf{g}_T \text{Re}\langle \Gamma_S \psi, \Gamma_T \psi \rangle$. Note that for any $v \in \mathbb{R}^{\mathcal{H}}$ we have $\langle v, Xv \rangle = \|\sum_S \mathbf{g}_S v_S \Gamma_S \psi\|^2 \geq 0$ and thus $X \succcurlyeq 0$. Furthermore, $\text{Tr}(X) = \sum_S \mathbf{g}_S^2 \|\Gamma_S \psi\|^2 = \|\mathbf{g}\|^2$, where $\|\Gamma_S \psi\|^2 = \|\psi\|^2 = 1$, since Γ_S is unitary.¹⁸ Finally, note that

$$\begin{aligned} \|H(\mathbf{g})\psi\|^2 &= \frac{1}{m} \left\langle \sum_{S \in \mathcal{H}} \mathbf{g}_S \Gamma_S \psi, \sum_{S \in \mathcal{H}} \mathbf{g}_S \Gamma_S \psi \right\rangle = \frac{1}{m} \sum_{S,T \in \mathcal{H}} \mathbf{g}_S \mathbf{g}_T \langle \Gamma_S \psi, \Gamma_T \psi \rangle \\ \implies \|H(\mathbf{g})\psi\|^2 &= \text{Re} \|H(\mathbf{g})\psi\|^2 = \frac{1}{m} \sum_{S,T \in \mathcal{H}} \mathbf{g}_S \mathbf{g}_T \text{Re}\langle \Gamma_S \psi, \Gamma_T \psi \rangle = \frac{1}{m} \sum_{S,T \in \mathcal{H}} X_{S,T} = \langle u, Xu \rangle, \end{aligned}$$

where recall that $u = u^{\mathcal{H}} := \mathbf{1}/\sqrt{m} \in \mathbb{R}^{\mathcal{H}}$.

Now, note that if $\varepsilon_{S,T} = -1$, i.e. $\Gamma_S \Gamma_T = -\Gamma_T \Gamma_S$, then $2 \text{Re}\langle \Gamma_S \psi, \Gamma_T \psi \rangle = \langle \Gamma_S \psi, \Gamma_T \psi \rangle + \langle \Gamma_T \psi, \Gamma_S \psi \rangle = \langle \psi, \Gamma_S \Gamma_T \psi \rangle + \langle \psi, \Gamma_T \Gamma_S \psi \rangle = 0$, i.e. $\varepsilon_{S,T} = -1 \implies X_{S,T} = 0$. Consequently,

$$\text{Tr}(\mathcal{E}X) = \frac{1}{m} \sum_{S,T} \varepsilon_{S,T} X_{S,T} = \frac{1}{m} \sum_{S,T} X_{S,T} = \|H(\mathbf{g})\psi\|^2.$$

On the other hand, by [Proposition 4.8](#) we have $\mathcal{E} \preccurlyeq (q + \delta)uu^* + (\rho^+ + \delta)(\Pi_{u^\perp})$. Since $X \succcurlyeq 0$, we have¹⁹

$$\text{Tr}(\mathcal{E}X) \leq \text{Tr}((q + \delta)uu^* + (\rho^+ + \delta)(\Pi_{u^\perp}))X.$$

But $\text{Tr}(uu^*X) = \langle u, Xu \rangle = \frac{1}{m} \sum_{S,T \in \mathcal{H}} X_{S,T} = \text{Tr}(\mathcal{E}X)$. Consequently,

$$\text{Tr}(\mathcal{E}X) \leq (q + \delta) \text{Tr}(\mathcal{E}X) + (\rho^+ + \delta)(\text{Tr}(X) - \text{Tr}(\mathcal{E}X)) = (q - \rho^+) \text{Tr}(\mathcal{E}X) + (\rho^+ + \delta) \text{Tr}(X)$$

$$\implies \|H(\mathbf{g})\psi\|^2 = \text{Tr}(\mathcal{E}X) \leq \frac{\rho^+ + \delta}{1 - q + \rho^+} \cdot \|\mathbf{g}\|^2 \implies \|H(\mathbf{g})\psi\| \leq \sqrt{\frac{\rho^+ + \delta}{1 - q}} \cdot \|\mathbf{g}\|,$$

as desired. $\square$

We now combine these results to obtain a lower bound on the operator norm of the SYK Hamiltonian in terms of the spectral edge of the twisted boson operator $x = a + a^*$ on the relevant Krylov subspace $\mathcal{R}$.

Lemma 4.18 (Lower Bounds on Operator Norm of the SYK Hamiltonian). *Let $n \geq k \geq 2$ be even integers, and let $\alpha := k^2/n < 1/16$. Consider the SYK Hamiltonian $H_{\text{SYK}}^{\mathcal{H}}$ (from [Eq. \(8\)](#)) for the hypergraph $\mathcal{H}$. Let $\mathcal{R}$ be any space of the kind defined in [Definition 4.11](#). Then*

$$\mathbb{E} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} \geq \lambda_{\max}(\Pi_{\mathcal{R}} x \Pi_{\mathcal{R}}) - O(1) \cdot \sqrt{\frac{L(\rho^+ + \delta)}{1 - q}}.$$

Proof. Note that $H_{\text{SYK}}^{\mathcal{H}} = H(\mathbf{g})$. Apply [Lemma 4.16](#) to obtain that there exists $F \in \mathcal{K}(\mathbb{R}^m; \mathbb{C}^{N \times N})$ such that

$$\mathbb{E}_{\mathbf{g} \sim \mathcal{N}(0, \text{Id}_m)} [\|H(\mathbf{g})\|_{\text{op}} \cdot w(\mathbf{g})] \geq \lambda_{\max}(\Pi_{\mathcal{R}} x \Pi_{\mathcal{R}}).$$

¹⁸ Recall that we fixed an unitary representation for $\{\gamma_i\}_{1 \leq i \leq n}$ into $\mathbb{C}^{N \times N}$.

¹⁹ Note that if $X \succcurlyeq 0$ and $A \succcurlyeq B$, then $\text{Tr}(BX) \leq \text{Tr}(AX)$, since the product of two PSD matrices has non-negative trace.

where $w(\mathbf{g}) := \|F(\mathbf{g})\|_{\text{HS}}^2$. Since $\mathbb{E}_{\mathbf{g}} w(\mathbf{g}) = 1$, we have

$$\begin{aligned} \mathbb{E}_{\mathbf{g}} [\|H(\mathbf{g})\|_{\text{op}} \cdot w(\mathbf{g})] &= \mathbb{E}_{\mathbf{g}} \|H(\mathbf{g})\|_{\text{op}} + \mathbb{E}_{\mathbf{g}} \left[\left(\|H(\mathbf{g})\|_{\text{op}} - \mathbb{E}_{\mathbf{g}} \|H(\mathbf{g})\|_{\text{op}} \right) \cdot w(\mathbf{g}) \right] \\ &\stackrel{\text{H\"older}}{\leq} \mathbb{E}_{\mathbf{g}} \|H(\mathbf{g})\|_{\text{op}} + \left\| \|H(\mathbf{g})\|_{\text{op}} - \mathbb{E}_{\mathbf{g}} \|H(\mathbf{g})\|_{\text{op}} \right\|_p \cdot \|w(\mathbf{g})\|_{p'}, \end{aligned}$$

where we set $p := L + 1$, where L is as in [Lemma 4.12](#). Now, note that

$$\|w(\mathbf{g})\|_{p'} = \|F(\mathbf{g})\|_{\mathcal{K}, 2p'}^2 \stackrel{\text{Proposition 3.8}}{\leq} (2p' - 1)^L \|F(\mathbf{g})\|_{\mathcal{K}, 2}^2 = (1 + 2/L)^L \cdot \mathbb{E}_{\mathbf{g}} w(\mathbf{g}) \leq O(1).$$

By [Fact 3.7](#) and [Lemma 4.17](#) we also have

$$\left\| \|H(\mathbf{g})\|_{\text{op}} - \mathbb{E}_{\mathbf{g}} \|H(\mathbf{g})\|_{\text{op}} \right\|_p \leq O(1) \cdot \sqrt{\frac{\rho^+ + \delta}{1 - q}} \cdot \sqrt{L},$$

as desired. $\square$

5 Operator Norm of the SYK Model

Recall the sign matrix $\mathcal{E} \in \mathbb{C}^{\mathcal{H} \times \mathcal{H}}$ defined as $\mathcal{E}_{S,T} := |\mathcal{H}|^{-1} \cdot \varepsilon_{S,T} = |\mathcal{H}|^{-1} \cdot (-1)^{|S \cap T|}$. Also recall the other notation from [Eq. \(13\)](#):

$$\begin{aligned} u &:= \mathbf{1} / \sqrt{|\mathcal{H}|}, \\ q &:= \langle u, \mathcal{E} u \rangle, \\ \delta &:= \|(\Pi_{u^\perp}) \mathcal{E} u\|, \\ \rho^+ &:= \lambda_{\max}((\Pi_{u^\perp}) \mathcal{E} (\Pi_{u^\perp})), \\ \rho^- &:= \max \{0, -\lambda_{\min}((\Pi_{u^\perp}) \mathcal{E} (\Pi_{u^\perp}))\}. \end{aligned}$$

We first bound these quantities in the “dense” case, i.e. the case when $\mathcal{H} = \binom{[n]}{k}$:

5.1 The Dense Case: $\mathcal{H} = \binom{[n]}{k}$

Throughout this section assume $\mathcal{H} = \binom{[n]}{k}$.

Recall the setting of [Fact 3.1](#), i.e. we have matrices $M_t \in \mathbb{C}^{\binom{[n]}{k} \times \binom{[n]}{k}}$ given as $M_t(S, T) := \binom{|S \cap T|}{t}$, and we have a detailed description of their spectrum. Now note that

$$(-1)^{|S \cap T|} = (1 - 2)^{|S \cap T|} = \sum_{t=0}^{|S \cap T|} (-2)^t \binom{|S \cap T|}{t} = \sum_{t=0}^k (-2)^t \binom{|S \cap T|}{t},$$

and consequently we have

$$\mathcal{E} = \frac{1}{\binom{n}{k}} \sum_{t=0}^k (-2)^t M_t,$$

i.e. the sign matrix $\mathcal{E}$ is a matrix in the Johnson scheme. By [Fact 3.1](#) we obtain that the eigenvalues of $\mathcal{E}$ are given by

$$\lambda_j := \binom{n}{k}^{-1} \cdot \sum_{t=j}^k (-2)^t \binom{k-j}{t-j} \cdot \binom{n-t-j}{k-t} \quad (27)$$

for $0 \leq j \leq k$, with λ_j occurring $d_j := \binom{n}{j} - \binom{n}{j-1}$ times in $\text{spec}(\mathcal{E})$. Furthermore, $u = \mathbf{1}/\sqrt{\binom{n}{k}}$ is the eigenvector corresponding to λ_0 . Consequently, in the context of Eq. (13) we obtain that

$$q = \lambda_0, \delta = 0, \rho^+ = \max_{j \geq 1} \lambda_j. \quad (28)$$

To upper bound these quantities, we need the following auxiliary technical lemma:

Lemma 5.1 (Sign Matrix Eigenvalues). *Suppose $n \geq k \geq 2$. Write $\alpha := k^2/n$, and assume $\alpha < 1/16$. For λ_j as in Eq. (27), we have*

$$\left| \frac{\lambda_j}{(-2)^j \binom{n-2j}{k-j} \binom{n}{k}^{-1}} - 1 \right| \leq 8\alpha. \quad (29)$$

In particular, $\lambda_j > 0$ if and only if j is even. Moreover,

- (1) $(1 - k/n)(1 - 8\alpha) \leq |\lambda_1|/(2k/n) \leq 1 + 8\alpha$.
- (2) $|\lambda_j| \leq 2(8k/n)^j$ for all $j \geq 0$. Consequently, $\sum_{j=0}^k d_j |\lambda_j| \leq e^{9k}$, where $d_j := \binom{n}{j} - \binom{n}{j-1}$.
- (3) $\max_{\substack{2 \leq j \leq k \\ j \text{ even}}} \lambda_j = \Theta(\alpha/n)$.
- (4) $\max_{\substack{1 \leq j \leq k \\ j \text{ odd}}} |\lambda_j| = |\lambda_1| \leq (2 + O(\alpha)) \cdot \sqrt{\alpha/n}$.
- (5) $2\alpha - 2\alpha^2 \leq 1 - \lambda_0 \leq 2\alpha$.

Proof. We know that

$$\lambda_j = \binom{n}{k}^{-1} \cdot \sum_{t=j}^k (-2)^t \binom{k-j}{t-j} \cdot \binom{n-t-j}{k-t} = \binom{n}{k}^{-1} (-2)^j \cdot \underbrace{\sum_{s=0}^{k-j} (-2)^s \binom{k-j}{s} \cdot \binom{n-2j-s}{k-j-s}}_{:=f(s)}.$$

For any $s \geq 0$, we have

$$\left| \frac{f(s)}{f(0)} \right| = \left| \frac{(-2)^s \binom{k-j}{s} \cdot \binom{n-2j-s}{k-j-s}}{\binom{n-2j}{k-j}} \right| \leq \frac{1}{s!} \left(\frac{2k^2}{n-2k} \right)^s = \frac{1}{s!} \left(\frac{2\alpha}{1-2k/n} \right)^s \leq \frac{(4\alpha)^s}{s!}.$$

Consequently,

$$\left| \frac{\lambda_j}{(-2)^j \binom{n-2j}{k-j} \binom{n}{k}^{-1}} - 1 \right| \leq \sum_{s=1}^{k-j} \frac{(4\alpha)^s}{s!} \leq e^{4\alpha} - 1 \leq 8\alpha,$$

as desired. Item 1 then follows by direct computation.

For Item 2, we have that

$$\binom{n-2j}{k-j} \binom{n}{k}^{-1} = \frac{(k)_j (n-k)_j}{(n)_{2j}} \leq \frac{k^j n^j}{(n-2k)^{2j}} \leq \left(\frac{4k}{n} \right)^j$$

where the last inequality holds since $n > 16k^2 \geq 4k$. Consequently,

$$|\lambda_j| \leq (1 + 8\alpha) \cdot 2^j \binom{n-2j}{k-j} \binom{n}{k}^{-1} \leq 2 \cdot \left(\frac{8k}{n} \right)^j,$$

as desired. Consequently, since $d_j \leq \binom{n}{j} \leq n^j/j!$, we have

$$\sum_{j=0}^k d_j |\lambda_j| \leq 2 \sum_{j=0}^k \frac{n^j}{j!} \cdot \left(\frac{8k}{n} \right)^j \leq 2 \sum_{j=0}^{\infty} \frac{(8k)^j}{j!} = 2e^{8k} \leq e^{9k}$$

since $k \geq 2$. [Item 3](#) is a direct consequence of [Eq. \(29\)](#) and [Item 2](#), and [Item 4](#) follows from [Item 1](#) and the fact that $\alpha < 1/16$.

Finally, let us consider λ_0 . Note that

$$\lambda_0 = \binom{n}{k}^{-1} \cdot \sum_{s=0}^k (-2)^s \binom{k}{s} \cdot \binom{n-s}{k-s} \stackrel{(*)}{=} \sum_{s=0}^k (-2)^s \mathbb{E} \binom{X}{s} = \mathbb{E} \sum_{s=0}^k (-2)^s \binom{X}{s} = \mathbb{E} (-1)^X,$$

where $X = |S \cap T|$ is the random variable obtained by considering the intersection size of the sets S, T independently and uniformly chosen from $\binom{[n]}{k}$. To see why $(*)$ is true, note that $\binom{X}{s} = \sum_{U \in \binom{S}{s}} \mathbf{1}(U \subseteq T)$, and thus $\mathbb{E} \left[\binom{X}{s} \mid S \right] = \sum_{U \in \binom{S}{s}} \Pr_T(U \subseteq T) = \binom{k}{s} \cdot \frac{\binom{n-s}{k-s}}{\binom{n}{k}}$. Since $\mathbb{E} \left[\binom{X}{s} \mid S \right]$ does not depend on S , we have $\mathbb{E} \binom{X}{s} = \binom{k}{s} \cdot \frac{\binom{n-s}{k-s}}{\binom{n}{k}}$, as desired.

Consequently, $1 - \lambda_0 = \mathbb{E}_{S,T}(1 - (-1)^X) = 2 \cdot \mathbb{E}_{S,T} \mathbf{1}(|S \cap T| \text{ is odd})$. Finally, note that for any integer $t \geq 0$, we have $t - t(t-1) \leq \mathbf{1}(t \text{ is odd}) \leq t$. Note that $\mathbb{E}_{S,T} |S \cap T| = \alpha$, $\mathbb{E}_{S,T} \binom{|S \cap T|}{2} = \binom{k}{2}^2 / \binom{n}{2} \leq \alpha^2/2$ for $\alpha \leq 1$. Thus

$$2(\alpha - \alpha^2) \leq 1 - \lambda_0 \leq 2\alpha,$$

as desired. $\square$

Putting [Eq. \(28\)](#) and [Lemma 5.1](#) yields an upper bound on the spectral norm of the SYK Hamiltonian for the dense case $\mathcal{H} = \binom{[n]}{k}$:

Theorem 5.2 (Operator Norm of the Dense SYK: Upper Bound). *Let $n \geq k \geq 2$ be even integers, and let $\alpha := k^2/n < 1/16$. Consider the SYK Hamiltonian (from [Eq. \(8\)](#)) for the dense case $\mathcal{H} = \binom{[n]}{k}$, i.e.*

$$H := \frac{1}{\sqrt{\binom{n}{k}}} \sum_{S \in \binom{[n]}{k}} g_S \Gamma_S$$

where $\{g_S\}_{S \in \binom{[n]}{k}}$ are i.i.d $\mathcal{N}(0, 1)$ Gaussians. Then

$$\mathbb{E} \|H\|_{\text{op}} \leq \sqrt{\frac{2}{\alpha}} + O(1).$$

In particular, if $\alpha \leq o_n(1)$ then $\mathbb{E} \|H\|_{\text{op}} \leq (1 + o(1)) \cdot \sqrt{2/\alpha}$.

Proof. Let $\ell \geq 1$ be an integer. Then

$$\begin{aligned} \mathbb{E} [\|H\|_{\text{op}}]^{2\ell} &\stackrel{\text{Jensen}}{\leq} \mathbb{E} [\|H\|_{\text{op}}^{2\ell}] \leq \mathbb{E} [\text{Tr}(H^{2\ell})] = N \cdot \mathbb{E} [\text{tr}(H^{2\ell})] \stackrel{\text{Proposition 4.7}}{=} N \cdot \varphi(x^{2\ell}) \\ &\stackrel{\text{Lemma 4.9}}{\leq} N \cdot \left(\frac{4(1 + (\rho^+ + \delta)\ell)}{1 - (q - \rho^+)_+} \right)^\ell \end{aligned}$$

where $N = 2^{n/2}$ is the dimension of H , and $\varphi, x, q, \rho^+, \delta$ are defined as in [Section 4](#). Consequently,

$$\mathbb{E} \|H\|_{\text{op}} \leq 2 \cdot 2^{n/(4\ell)} \cdot \sqrt{\frac{1 + (\rho^+ + \delta)\ell}{1 - (q - \rho^+)_+}}$$

for all integers $\ell \geq 1$.

Now, by Eq. (28) we know that $q = \lambda_0, \delta = 0, \rho^+ = \max_{j \geq 1} \lambda_j$. Note that $q - \rho^+ \leq q \leq 1 - 2\alpha + 2\alpha^2 < 1$, and thus by Lemma 5.1 we obtain that $1 - q \geq 2(\alpha - \alpha^2), \rho^+ \leq O(\alpha/n)$. Consequently, $1 - (q - \rho^+)_+ = 1 - q + \rho^+ \geq 1 - q$ for large enough n ,²⁰ and thus

$$\mathbb{E} \|H\|_{\text{op}} \leq 2^{n/(4\ell)} \cdot \sqrt{\frac{2 + O(\alpha\ell/n)}{\alpha - \alpha^2}}.$$

Choose $\ell := \lceil n/\sqrt{\alpha} \rceil$ to obtain that

$$\mathbb{E} \|H\|_{\text{op}} \leq \sqrt{\frac{2}{\alpha}} \cdot (1 + O(\sqrt{\alpha})) \leq \sqrt{\frac{2}{\alpha}} + O(1),$$

as desired. $\square$

Theorem 5.3 (Operator Norm of the Dense SYK: Lower Bound). *Let $n \geq k \geq 2$ be even integers, and let $\alpha := k^2/n < 1/16$. Consider the SYK Hamiltonian (from Eq. (8)) for the dense case $\mathcal{H} = \binom{[n]}{k}$. Then*

$$\mathbb{E} \|H\|_{\text{op}} \geq \sqrt{\frac{2}{\alpha}} \cdot \left(1 - O(1) \cdot \max\left\{k^{-1/2}, \alpha^2\right\}\right) = \frac{\sqrt{2n}}{k} \cdot \left(1 - O(1) \cdot \max\left\{k^{-1/2}, \frac{k^4}{n^2}\right\}\right).$$

Proof. Throughout the proof we'll borrow notation from Lemma 4.12. By Lemma 4.18 we know that if $B \geq \Omega(1)$, then

$$\mathbb{E} \|H\|_{\text{op}} \geq \lambda_{\max}(\Pi_{\mathcal{R}} \times \Pi_{\mathcal{R}}) - O(1) \cdot \sqrt{\frac{L(\rho^+ + \delta)}{1 - q}}.$$

Now, we'll use Lemma 5.1 to compute the parameters in Lemma 4.12. Note that $\rho^- = |\lambda_1| \leq O(\sqrt{\alpha/n})$, and thus $B = \sqrt{\nu/\rho^-} \geq \Omega\left(\sqrt{\alpha/\sqrt{\alpha/n}}\right) \geq \Omega(\sqrt{k}) \geq \Omega(1)$, and thus our invocation of Lemma 4.12 is justified. Choose $\kappa = \Theta(\sqrt{k}) \leq B$. Then $L \leq O(\kappa/\nu) \leq O(n/k^{3/2}), 1 - q \geq \Omega(k^2/n), \rho^+ + \delta \leq O(k^2/n^2)$, and thus

$$\frac{L(\rho^+ + \delta)}{1 - q} \leq O(1) \cdot \frac{n \cdot k^2 \cdot n}{k^{3/2} \cdot n^2 \cdot k^2} \leq O(k^{-3/2}).$$

Also

$$\lambda_{\max}(\Pi_{\mathcal{R}} \times \Pi_{\mathcal{R}}) \geq \sqrt{\frac{2}{\alpha}} \cdot \left(1 - O(1) \cdot \max\left\{k^{-1/2}, \alpha^2\right\}\right),$$

and we're done. $\square$

Finally, using Fact 3.7, we can even obtain strong subgaussian concentration for $\|H_{\text{SYK}}\|_{\text{op}}$:

Theorem 5.4 (Subgaussian Tails for the Operator Norm). *Let $n \geq k \geq 2$ be even integers, and let $\alpha := k^2/n < 1/16$. Consider the SYK Hamiltonian (from Eq. (8)) for the dense case $\mathcal{H} = \binom{[n]}{k}$. Then for any $t \geq 0$ we have*

$$\Pr\left(\left|\|H\|_{\text{op}} - \mathbb{E}_{\mathcal{g}} \|H\|_{\text{op}}\right| \geq t\right) \leq 2 \cdot \exp(-\Omega(nt^2)).$$

Proof. Towards applying Fact 3.7, consider the function $\mathbf{g} \mapsto \|H(\mathbf{g})\|_{\text{op}} = \|H\|_{\text{op}}$. By Lemma 4.17 we know that the Lipschitz constant of this map is $\leq \sqrt{(\rho^+ + \delta)/(1 - q)}$. From the proof of Theorem 5.3 we know that $(\rho^+ + \delta)/(1 - q) \leq O(1/n)$. We are now done by Fact 3.7. $\square$

²⁰ Note that we claim the result for all $n > 16k^2$. For n smaller than what we need for our inequalities to hold, we can simply inflate the implicit constant in the $O(1)$ in the theorem statement to make it work.

5.2 The Sparse Case: $\mathcal{H}$ is a random hypergraph

Suppose $\mathcal{H}$ is a random k -uniform hypergraph with m hyperedges, i.e. every hyperedge of $\mathcal{H}$ sampled independently, uniformly at random, from $\binom{[n]}{k}$. Note that $\mathcal{H}$ is a multiset in general.

Write $\mathcal{E}^{\text{comp}} := \mathcal{E}^{\binom{[n]}{k}} = \binom{n}{k}^{-1} \cdot (\varepsilon_{S,T})_{S,T \in \binom{[n]}{k}}$. Then note that

$$\mathcal{E}^{\mathcal{H}} = \frac{\binom{n}{k}}{m} P^* \mathcal{E}^{\text{comp}} P,$$

where $P : \mathbb{C}^{\mathcal{H}} \rightarrow \mathbb{C}^{\binom{[n]}{k}}$ is the usual map where for every $S \in \mathcal{H}$ we have $Pe_S := e_S \in \mathbb{C}^{\binom{[n]}{k}}$, where $\{e_S\}_{S \in \mathcal{H}}$ (resp. $\{e_S\}_{S \in \binom{[n]}{k}}$) is the standard basis for $\mathbb{C}^{\mathcal{H}}$ (resp. $\mathbb{C}^{\binom{[n]}{k}}$).

Furthermore, since $\mathcal{E}^{\text{comp}}$ is Hermitian, we can write

$$\mathcal{E}^{\text{comp}} = \frac{\lambda_0 \mathbf{1}\mathbf{1}^*}{\binom{n}{k}} + \sum_{j \geq 1} \lambda_j \Pi_j$$

where λ_j are as in Eq. (27), and Π_j refers to the orthogonal projection $\mathbb{C}^{\binom{[n]}{k}} \rightarrow V_j$, where V_j is as in Fact 3.1. Write

$$G^+ := \sum_{j \geq 1: \lambda_j \geq 0} \lambda_j \Pi_j, \quad G^- := - \sum_{j \geq 1: \lambda_j < 0} \lambda_j \Pi_j,$$

and thus $\mathcal{E}^{\text{comp}} = \frac{\lambda_0 \mathbf{1}\mathbf{1}^*}{\binom{n}{k}} + G^+ - G^-$.

Note that $G^+, G^- \succcurlyeq 0$. Furthermore by the $\mathfrak{S}_n$ -invariance of V_j (see Fact 3.1), we have that for any $S, T \in \binom{[n]}{k}$ we have $\langle e_S, G^+ e_S \rangle = \langle e_T, G^+ e_T \rangle$ for all $S, T \in \binom{[n]}{k}$. Since $\sum_{S \in \binom{[n]}{k}} \langle e_S, G^+ e_S \rangle = \text{Tr}(G^+)$ we obtain that $\langle e_S, G^+ e_S \rangle = \text{tr}(G^+)$ for all $S \in \binom{[n]}{k}$. Similarly, we also have $\langle e_S, G^- e_S \rangle = \text{tr}(G^-)$ for all $S \in \binom{[n]}{k}$.

Now write $A^\pm := \frac{\binom{n}{k}}{m} P^* G^\pm P$. Note that $A^\pm \succcurlyeq 0$.

Theorem 5.5 (Operator Norm of the Sparse SYK: Upper Bound). *Let $n \geq k \geq 2$ be even integers, and suppose $\alpha := k^2/n < 1/16$. Let $\mathcal{H}$ be a random k -uniform hypergraph with $m \geq 2^{Ck} n \log n$ edges, where $C > 0$ is some sufficiently large absolute constant. Let*

$$H_{\text{SYK}}^{\mathcal{H}} := \frac{1}{\sqrt{|\mathcal{H}|}} \sum_{S \in \mathcal{H}} g_S \Gamma_S$$

be the SYK Hamiltonian corresponding to $\mathcal{H}$ (as in Eq. (8)). Then with probability $\geq 1 - O(1/\log(n))$ over the randomness of $\mathcal{H}$ we have

$$\mathbb{E}_g \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} \leq \sqrt{\frac{2}{\alpha}} \cdot \left(1 + O\left(\sqrt{\alpha + e^{-\Omega(k)}}\right)\right) = \frac{\sqrt{2n}}{k} \cdot \left(1 + O\left(\sqrt{\frac{k^2}{n} + e^{-\Omega(k)}}\right)\right).$$

In particular if $k \geq \omega(\log(n))$ then $\mathbb{E} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} \leq \sqrt{2n}/k + O(1)$.

Proof. Define q, δ, ρ^+ as in Eq. (13). Note that

$$\mathcal{E}^{\mathcal{H}} = \lambda_0 u u^* + A^+ - A^-,$$

where $u := \mathbf{1}/\sqrt{m}$. Let $\Pi = \Pi_{u^\perp}$, and note that

$$\rho^+ = \lambda_{\max}(\Pi \mathcal{E}^{\mathcal{H}} \Pi) = \lambda_{\max}(\Pi(A^+ - A^-)\Pi).$$

Since $\Pi(A^+ - A^-)\Pi \preceq \Pi A^+ \Pi$, and since $\|\Pi A^+ \Pi\|_{\text{op}} \leq \|A^+\|_{\text{op}}$, we have $\rho^+ \leq \lambda_{\max}(A^+) = \|A^+\|_{\text{op}}$. To bound $\|A^+\|_{\text{op}}$ we aim to apply [Lemma 3.3](#). Towards that, note that $\text{Tr}(G^+) = \sum_{j \geq 1: \lambda_j \geq 0} \lambda_j \dim(V_j) \leq \sum_{j \geq 1} |\lambda_j| \dim(V_j)$. Since $\alpha < 1/16$ we obtain by [Lemma 5.1](#) that $\text{Tr}(G^+) \leq e^{9k}$. Also by [Lemma 5.1](#) we have that $\mu := \|G^+\|_{\text{op}} = \max_{j \geq 1} \lambda_j = \max_{j \geq 2, j \text{ even}} \lambda_j = \Theta(\alpha/n) = \Theta(k^2/n^2)$. Thus write $d := \text{Tr}(G)/\mu \leq O(e^{9k}/(k^2/n^2)) \leq O(e^{9k}n^2) \leq e^{O(k)}n^2$. Finally, since $\langle e_S, G^+ e_S \rangle = \text{tr}(G^+)$ for all $S \in \binom{[n]}{k}$, [Lemma 3.3](#) applies and we obtain that with probability $\geq 1 - O(n^{-200})$,²¹

$$\begin{aligned} \|A^+\|_{\text{op}} &\leq O(1) \cdot \max \left\{ O\left(\frac{\alpha}{n}\right), \frac{e^{9k}}{m} \cdot \log(e^{O(k)}n^2) \right\} \leq O(1) \cdot \max \left\{ \frac{\alpha}{n}, \frac{e^{9k}}{m} \cdot (k + \log n) \right\} \\ &\leq O(1) \cdot \max \left\{ \frac{\alpha}{n}, \frac{e^{O(k)} \cdot \log(n)}{m} \right\} \implies \rho^+ \leq O(1) \cdot \max \left\{ \frac{\alpha}{n}, \frac{e^{O(k)} \cdot \log(n)}{m} \right\}. \end{aligned}$$

Now write $\tilde{A} := A^+ - A^-$. Note that $\tilde{A}(S, T) = m^{-1}((-1)^{|S \cap T|} - \lambda_0)$. Now write $\tilde{a}_{ij} := (-1)^{|S_i \cap S_j|} - \lambda_0$, where recall that $S_1, \dots, S_m$ are the hyperedges of $\mathcal{H}$, i.e they are independent samples from $\binom{[n]}{k}$. Note that by the proof of [Lemma 5.1](#) we know that for $i \neq j$, $\mathbb{E}_{\mathcal{H}} \tilde{a}_{ij} = 0$, $\mathbb{E}_{\mathcal{H}} \tilde{a}_{ij}^2 = 1 - \lambda_0^2$. Also note that $\tilde{a}_{ii} = 1 - \lambda_0$ with probability 1. Now suppose we have i, j, i', j' with $i \neq j, i' \neq j'$. Then if $\#\{i, j, i', j'\} = 4$ then $\tilde{a}_{ij}, \tilde{a}_{i'j'}$ are independent mean 0 random variables, and thus $\mathbb{E}_{\mathcal{H}}[\tilde{a}_{ij}\tilde{a}_{i'j'}] = 0$. Suppose $\#\{i, j, i', j'\} = 3$. WLOG $i = i'$. Then conditional on S_i , $h_{ij}, h_{ij'}$ are still independent, and thus we still have $\mathbb{E}_{\mathcal{H}}[\tilde{a}_{ij}\tilde{a}_{ij'}] = 0$. Consequently, $\mathbb{E}_{\mathcal{H}}[\tilde{a}_{ij}\tilde{a}_{i'j'}] \neq 0$ only if $\{i, j\} = \{i', j'\}$ in which case it equals $1 - \lambda_0^2$.

Note that $\delta = \|(\Pi_{u^\perp})\tilde{A}u\|_2 = \|\tilde{A}u - uu^*\tilde{A}u\|_2$, and thus $\delta^2 = \|\tilde{A}u\|_2^2 - |u^*\tilde{A}u|^2$. Similarly, $q = u^*\mathcal{E}^{\mathcal{H}}u = \lambda_0 + m^{-2} \sum_{1 \leq i, j \leq m} \tilde{a}_{ij} = \lambda_0 + m^{-1} \cdot (1 - \lambda_0) + 2m^{-2} \sum_{1 \leq i < j \leq m} \tilde{a}_{ij}$.

Thus $\mathbb{E}_{\mathcal{H}} q = \lambda_0 + m^{-1} \cdot (1 - \lambda_0)$. Furthermore,

$$\begin{aligned} \mathbb{E}_{\mathcal{H}} \left[\left(q - \lambda_0 - \frac{1 - \lambda_0}{m} \right)^2 \right] &= \frac{4}{m^4} \sum_{\substack{1 \leq i < j \leq m \\ 1 \leq i' < j' \leq m}} \mathbb{E}_{\mathcal{H}}[\tilde{a}_{ij}\tilde{a}_{i'j'}] = \frac{4}{m^4} \sum_{1 \leq i < j \leq m} \mathbb{E}_{\mathcal{H}}[\tilde{a}_{ij}^2] = \frac{4}{m^4} \binom{m}{2} (1 - \lambda_0^2) \\ \implies \mathbb{E}_{\mathcal{H}} [(q - \lambda_0)^2] &= \frac{(1 - \lambda_0)^2}{m^2} + \frac{2(m-1)}{m^3} (1 - \lambda_0^2) \leq \frac{4(1 - \lambda_0)}{m^2} \stackrel{\text{Lemma 5.1}}{\leq} \frac{8\alpha}{m^2}. \end{aligned}$$

Thus if we write $\zeta := 1/(100 \log n)$, then by Markov's inequality, with probability $\geq 1 - \zeta$, we have $|q - \lambda_0| \leq O(m^{-1} \sqrt{\alpha/\zeta})$.

Now, note that $\mathbb{E}_{\mathcal{H}} \delta^2 \leq \mathbb{E}_{\mathcal{H}} \|\tilde{A}u\|_2^2$. Thus we estimate $\mathbb{E}_{\mathcal{H}} \|\tilde{A}u\|_2^2$. Towards that, note that

$$\mathbb{E}_{\mathcal{H}} \|\tilde{A}u\|_2^2 = \sum_{i=1}^m \mathbb{E}_{\mathcal{H}} (\tilde{A}u)_i^2 = \frac{1}{m^3} \sum_{i=1}^m \mathbb{E}_{\mathcal{H}} \left[\left(\sum_{j=1}^m \tilde{a}_{ij} \right)^2 \right].$$

Fix any i and any choice for S_i . Write $\eta := 1 - \lambda_0$. Then

$$\begin{aligned} \mathbb{E}_{\mathcal{H}} \left[\left(\sum_{j=1}^m \tilde{a}_{ij} \right)^2 \mid S_i \right] &= \mathbb{E}_{\mathcal{H}} \left[\left(\eta + \sum_{j \neq i} \tilde{a}_{ij} \right)^2 \mid S_i \right] = \eta^2 + 2\eta \mathbb{E}_{\mathcal{H}} \left[\sum_{j \neq i} \tilde{a}_{ij} \mid S_i \right] + \mathbb{E}_{\mathcal{H}} \left[\left(\sum_{j \neq i} \tilde{a}_{ij} \right)^2 \mid S_i \right] \\ &= \eta^2 + \mathbb{E}_{\mathcal{H}} \left[\left(\sum_{j \neq i} \tilde{a}_{ij} \right)^2 \mid S_i \right] = \eta^2 + \mathbb{E}_{\mathcal{H}} \left[\sum_{j \neq i} \tilde{a}_{ij}^2 \mid S_i \right] + 2 \cdot \mathbb{E}_{\mathcal{H}} \left[\sum_{\substack{j, j' \in [m] \setminus \{i\} \\ j < j'}} \tilde{a}_{ij} \tilde{a}_{ij'} \mid S_i \right] \end{aligned}$$

²¹ Note that we also have $d = \text{Tr}(G^+)/\mu \geq \Omega(k^2/(k^2/n^2)) \geq \Omega(n^2)$.

$$= \eta^2 + (m-1)(1-\lambda_0^2),$$

where we use the fact that $\mathbb{E}[\tilde{a}_{ij} \mid S_i] = \mathbb{E}[\tilde{a}_{ij}\tilde{a}_{ij'} \mid S_i] = 0$ for $j, j' \neq i, j \neq j'$. Consequently

$$\mathbb{E}_{\mathcal{H}} \|\tilde{A}u\|_2^2 = \frac{(1-\lambda_0)^2 + (m-1)(1-\lambda_0^2)}{m^2} \leq \frac{2(1-\lambda_0)}{m} \leq \frac{4\alpha}{m} \implies \mathbb{E}_{\mathcal{H}} \delta^2 \leq \frac{4\alpha}{m}.$$

As above, by Markov's inequality, with probability $\geq 1 - \zeta$ we have $\delta \leq O(\sqrt{\alpha/(m\zeta)})$.

Consequently, with probability $\geq 1 - O(1/\log n)$, we have that

$$\begin{aligned} \rho^+ + \delta &\leq O\left(\frac{\alpha}{n}\right) + \frac{e^{O(k)} \cdot \log(n)}{m} + O\left(\sqrt{\frac{\alpha \log n}{m}}\right) \leq O\left(\frac{\alpha}{n}\right) + \frac{e^{-\Omega(k)}}{n} + O\left(\frac{k}{2^{Ck/2n}}\right) \\ &\leq O(1) \cdot \underbrace{\left(\frac{\alpha}{n} + \frac{e^{-\Omega(k)}}{n}\right)}_{:=\xi/n}, \end{aligned}$$

where we use the fact that $m \geq 2^{Ck} n \log(n)$ for some sufficiently large absolute constant $C > 0$. With probability $\geq 1 - O(1/\log n)$ we also have that

$$q - \rho^+ \geq \lambda_0 - O\left(\frac{\sqrt{\alpha \log(n)}}{m}\right) - O\left(\frac{\alpha}{n}\right) - \frac{e^{-\Omega(k)}}{n} \stackrel{\text{Lemma 5.1}}{\geq} \frac{7}{8} - O\left(\frac{1}{n}\right) > 0$$

for sufficiently large n . Consequently, $(q - \rho^+)_+ = q - \rho^+$. Furthermore,

$$1 - (q - \rho^+)_+ = 1 - (q - \rho^+) \geq 1 - q \geq 1 - \lambda_0 - O\left(\frac{\sqrt{\alpha \log(n)}}{m}\right) \geq 2(\alpha - \alpha^2) - O\left(\frac{e^{-\Omega(k)}}{\sqrt{n^3 \log(n)}}\right) > 0$$

for sufficiently large n , since $\alpha \geq 4/n$.

Consequently, as in the proof of [Theorem 5.2](#), if $H_{\text{SYK}}^{\mathcal{H}} = \frac{1}{\sqrt{m}} \sum_{S \in \mathcal{H}} g_S \Gamma_S$ is the SYK Hamiltonian corresponding to the hypergraph $\mathcal{H}$, then we have (by [Lemma 4.9](#))

$$\mathbb{E}_{\{g_S\}} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} \leq 2 \cdot 2^{n/(4\ell)} \cdot \sqrt{\frac{1 + (\rho^+ + \delta)\ell}{1 - (q - \rho^+)_+}} \leq 2 \cdot 2^{n/(4\ell)} \cdot \sqrt{\frac{1 + (\rho^+ + \delta)\ell}{1 - q}}.$$

Thus with probability $\geq 1 - O(1/\log n)$ over the randomness of $\mathcal{H}$ we have that

$$\begin{aligned} \mathbb{E}_{\{g_S\}} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} &\leq 2 \cdot 2^{n/(4\ell)} \cdot \sqrt{\frac{1 + \xi\ell/n}{1 - q}} \leq 2 \cdot 2^{n/(4\ell)} \cdot \sqrt{\frac{1 + \xi\ell/n}{2(\alpha - \alpha^2) - O\left(\frac{e^{-\Omega(k)}}{\sqrt{n^3 \log(n)}}\right)}} \\ &\leq 2^{n/(4\ell)} \cdot \sqrt{\frac{2 + O(\xi\ell/n)}{(\alpha - \alpha^2) - \alpha \cdot O(e^{-\Omega(k)})/\sqrt{n \log(n)}}} \end{aligned}$$

Now set $\ell := \lceil n/\sqrt{\xi} \rceil$ to obtain that

$$\mathbb{E} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} \leq \sqrt{\frac{2}{\alpha - \alpha^2 - o_n(\alpha)}} \cdot (1 + O(\sqrt{\xi})) \leq \sqrt{\frac{2}{\alpha}} \cdot (1 + O(\sqrt{\xi})) \cdot \left(1 + O\left(\alpha + \frac{e^{-\Omega(k)}}{\sqrt{n \log(n)}}\right)\right).$$

Note that $\xi \leq O(\alpha + e^{-\Omega(k)})$, and thus

$$\mathbb{E} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} \leq \sqrt{\frac{2}{\alpha}} \cdot (1 + O(\sqrt{\alpha + e^{-\Omega(k)}})) \cdot \left(1 + O\left(\alpha + \frac{e^{-\Omega(k)}}{\sqrt{n \log(n)}}\right)\right)$$

$$\leq \sqrt{\frac{2}{\alpha}} \cdot \left(1 + O\left(\sqrt{\alpha + e^{-\Omega(k)}}\right)\right),$$

as desired. $\square$

Finally, we prove the corresponding lower bound for the sparse SYK.

Theorem 5.6 (Operator Norm of the Sparse SYK: Lower Bound). *Let $n \geq k \geq 2$ be even integers, and let $\alpha := k^2/n < 1/16$. Consider the SYK Hamiltonian (from Eq. (8)) for a random hypergraph $\mathcal{H}$ with $m \geq 2^{Ck} n \log(n)$ hyperedges for some large enough absolute constant $C > 0$. Then*

$$\mathbb{E} \|H\|_{\text{op}} \geq \sqrt{\frac{2}{\alpha}} \left(1 - O(1) \max\left\{k^{-1/4}, \alpha^2\right\}\right).$$

with probability $\geq 1 - O(1/\log n)$ over the randomness of $\mathcal{H}$.

Proof. Throughout the proof we'll borrow notation from Lemma 4.12. By Lemma 4.18 we know that if $B \geq \Omega(1)$, then

$$\mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} \geq \lambda_{\max}(\Pi_{\mathcal{R}} \times \Pi_{\mathcal{R}}) - O(1) \sqrt{\frac{L(\rho^+ + \delta)}{1 - q}}. \quad (30)$$

Note that for any unit vector $v \in \mathbb{R}^{\mathcal{H}}$ such that $\langle v, u \rangle = 0$, we have

$$-\langle v, \mathcal{E}^{\mathcal{H}} v \rangle = \langle v, A^- v \rangle - \langle v, A^+ v \rangle \leq \langle v, A^- v \rangle \leq \|A^-\|_{\text{op}} \implies \rho^- \leq \|A^-\|_{\text{op}}.$$

We now apply Lemma 3.3 to G^- . By Lemma 5.1,

$$\|G^-\|_{\text{op}} = \max_{\substack{j \geq 1 \\ \lambda_j < 0}} |\lambda_j| = |\lambda_1| \leq O\left(\sqrt{\frac{\alpha}{n}}\right) = O\left(\frac{k}{n}\right), \quad \text{Tr}(G^-) = \sum_{\substack{j \geq 1 \\ \lambda_j < 0}} |\lambda_j| \dim(V_j) \leq e^{9k}.$$

Moreover, by the $\mathfrak{S}_n$ -invariance of the Johnson eigenspaces, the diagonal entries of G^- are all equal. Hence Lemma 3.3 gives, with probability at least $1 - O(n^{-100})$,

$$\|A^-\|_{\text{op}} \leq O(1) \max\left\{\frac{k}{n}, \frac{e^{9k}}{m} \log\left(e^{O(k)} n^2\right)\right\} \leq O(1) \max\left\{\frac{k}{n}, \frac{e^{O(k)}(k + \log n)}{m}\right\} \leq O\left(\frac{k}{n}\right).$$

where the last inequality follows from $m \geq 2^{Ck} n \log n$ after choosing C sufficiently large. Consequently we have

$$\rho^- \leq O\left(\frac{k}{n}\right) = O\left(\sqrt{\frac{\alpha}{n}}\right).$$

As proved in Theorem 5.5, with probability at least $1 - O(1/\log n)$,

$$\delta \leq O\left(\sqrt{\frac{\alpha \log n}{m}}\right) \leq O\left(\frac{e^{-\Omega(k)}}{n}\right), \quad \rho^+ + \delta \leq O(1) \cdot \frac{\alpha + e^{-\Omega(k)}}{n} \leq O\left(\frac{k}{n}\right),$$

$$(2 + o_n(1))\alpha \geq 2\alpha + O\left(\frac{e^{-\Omega(k)}}{\sqrt{n^3 \log n}}\right) \geq \nu = 1 - q \geq 2(\alpha - \alpha^2) - O\left(\frac{e^{-\Omega(k)}}{\sqrt{n^3 \log n}}\right) \geq \Omega(\alpha).$$

Consequently we have $\nu = \Theta(\alpha)$. Thus

$$B := \sqrt{\frac{\nu}{\rho^- + \delta}} \geq \Omega\left(\sqrt{\frac{\alpha}{k/n}}\right) \geq \Omega(\sqrt{k}) \geq \Omega(1).$$

Hence the hypothesis needed by [Lemma 4.12](#) is satisfied. Now choose $\kappa = \Theta(\sqrt{k}) \leq B$, and note that this choice yields

$$L \leq O\left(\frac{\kappa}{\nu}\right) \leq O\left(\frac{n}{k^{3/2}}\right).$$

Thus

$$\frac{L(\rho^+ + \delta)}{1 - q} \leq O\left(\frac{n}{k^{3/2}}\right) \frac{(\alpha + e^{-\Omega(k)})/n}{\alpha} = O\left(\frac{\alpha + e^{-\Omega(k)}}{\alpha k^{3/2}}\right) \leq O\left(\frac{1}{\alpha \sqrt{k}}\right),$$

where in the last inequality we used $\alpha + e^{-\Omega(k)} \leq O(k)$ for $k \geq 2$. Therefore

$$\sqrt{\frac{L(\rho^+ + \delta)}{1 - q}} \leq O\left(\frac{1}{\sqrt{\alpha} k^{1/4}}\right) = \sqrt{\frac{2}{\alpha}} O(k^{-1/4}). \quad (31)$$

Finally [Lemma 4.12](#) applied with the preceding bounds on ν , $\rho^- + \delta$, κ , and L , gives

$$\lambda_{\max}(\Pi_{\mathcal{R}} \times \Pi_{\mathcal{R}}) \geq \sqrt{\frac{2}{\alpha}} \left(1 - O(1) \max\{k^{-1/2}, \alpha^2\}\right). \quad (32)$$

Combining (30), (31), and (32), and absorbing the transfer error into the existing $O(k^{-1/4})$ relative error, yields

$$\mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} \geq \sqrt{\frac{2}{\alpha}} \left(1 - O(1) \max\{k^{-1/4}, \alpha^2\}\right).$$

All the required inequalities hold simultaneously with probability at least $1 - O(1/\log n)$, and thus we are done. $\square$

Finally, as in [Theorem 5.4](#), we obtain strong subgaussian concentration for $\|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}}$:

Theorem 5.7 (Subgaussian Tails for the Operator Norm for Sparse SYK). *Let $n \geq k \geq 2$ be even integers, and let $\alpha := k^2/n < 1/16$. Consider the SYK Hamiltonian (from [Eq. \(8\)](#)) for a random hypergraph $\mathcal{H}$ with $m \geq 2^{Ck} n \log(n)$ hyperedges for some large enough absolute constant $C > 0$. Then for any $t \geq 0$ we have*

$$\Pr\left(\left|\|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}} - \mathbb{E}_{\mathbf{g}} \|H_{\text{SYK}}^{\mathcal{H}}\|_{\text{op}}\right| \geq t\right) \leq 2 \cdot \exp(-\Omega(kt^2))$$

with probability $\geq 1 - O(1/\log(n))$ over the randomness of $\mathcal{H}$.

Proof. Towards applying [Fact 3.7](#), consider the function $\mathbf{g} \mapsto \|H(\mathbf{g})\|_{\text{op}} = \|H\|_{\text{op}}$. By [Lemma 4.17](#) and the proof of [Theorem 5.3](#) we know that the Lipschitz constant of this map is $\leq \sqrt{(\rho^+ + \delta)/(1 - q)} \leq O(1/\sqrt{k})$, and we're now done by [Fact 3.7](#). $\square$

6 Acknowledgments

S.M. acknowledges support from a Brown Investigator Award, a program of the Brown Institute for Basic Sciences at the California Institute of Technology. S.M. thanks Dima Abanin for helpful discussions on quantum simulation and related collaborations.

Use of AI Tools: We observed (see [Section 2.1](#) for a key observation) that there is an explicit deterministic operator with a Johnson-like structure that captures the expected trace moments of H_{SYK} . We analyzed a preliminary version of this deterministic operator with the help of GPT-5.5 Pro and GPT-5.6 Pro. In particular, GPT-5.6 Pro suggested [Lemma 4.6](#), [Proposition 4.14](#), and [Lemma 4.15](#) to us. We also used GPT-5.6 Pro to search for related literature and assist with the verification of our technical proofs.

References

- [ACKK25] Eric R. Anschuetz, Chi-Fang Chen, Bobak T. Kiani, and Robbie King. Strongly Interacting Fermions Are Nontrivial yet Nonglassy. *Physical Review Letters*, 135(3):030602, July 2025. doi:[10.1103/cbqf-d24r](#). (pg. [2](#), [30](#))
- [BCD24] Joao Basso, Chi-Fang Chen, and Alexander M Dalzell. Optimizing random local hamiltonians by dissipation. *arXiv preprint arXiv:2411.02578*, 2024. (pg. [1](#), [2](#), [4](#), [5](#))
- [BF71] Oriol Bohigas and Jorge Flores. Two-body random hamiltonian and level density. *Physics Letters B*, 34(4):261–263, 1971. (pg. [2](#))
- [BINT19] Micha Berkooz, Mikhail Isachenkov, Vladimir Narovlansky, and Genis Torrents. Towards a full solution of the large n double-scaled syk model. *Journal of High Energy Physics*, 2019(3):1–72, 2019. (pg. [2](#), [11](#), [12](#))
- [BK02] Sergey B. Bravyi and Alexei Yu. Kitaev. Fermionic quantum computation. *Annals of Physics*, 298(1):210–226, 2002. URL: <https://www.sciencedirect.com/science/article/pii/S0003491602962548>, doi:[10.1006/aphy.2002.6254](#). (pg. [5](#), [14](#))
- [BKS97] Marek Bożejko, Burkhard Kümmerner, and Roland Speicher. q-gaussian processes: non-commutative and classical aspects. *Communications in Mathematical Physics*, 185(1):129–154, 1997. (pg. [12](#))
- [BM25] Micha Berkooz and Ohad Mamroud. A cordial introduction to double scaled syk. *Reports on Progress in Physics*, 88(3):036001, 2025. (pg. [2](#), [8](#), [9](#), [11](#), [12](#))
- [BNS18] Micha Berkooz, Prithvi Narayan, and Joan Simon. Chord diagrams, exact correlators in spin glasses and black hole bulk reconstruction. *Journal of High Energy Physics*, 2018(8):1–39, 2018. (pg. [11](#), [12](#))
- [BS91] Marek Bożejko and Roland Speicher. An example of a generalized brownian motion. *Communications in Mathematical Physics*, 137(3):519–531, 1991. (pg. [12](#))
- [BS94] Marek Bożejko and Roland Speicher. Completely positive maps on coxeter groups, deformed commutation relations, and operator spaces. *arXiv preprint funct-an/9408002*, 1994. (pg. [12](#))
- [CGPS22] Debanjan Chowdhury, Antoine Georges, Olivier Parcollet, and Subir Sachdev. Sachdev-ye-kitaev models and beyond: Window into non-fermi liquids. *Rev. Mod. Phys.*, 94:035004, Sep 2022. URL: <https://link.aps.org/doi/10.1103/RevModPhys.94.035004>, doi:[10.1103/RevModPhys.94.035004](#). (pg. [2](#))
- [Del73] Philippe Delsarte. An algebraic approach to the association schemes of coding theory. *Philips Res. Rep. Suppl.*, 10:vi+–97, 1973. (pg. [10](#), [15](#))
- [DKKA26] Matthew Ding, Robbie King, Bobak T. Kiani, and Eric R. Anschuetz. Optimizing Sparse SYK. *Quantum*, 10:2029, March 2026. [arXiv:2506.09037](#), doi:[10.22331/q-2026-03-16-2029](#). (pg. [2](#), [3](#), [4](#))

- [DMM94] Miroslav Dorešić, Stjepan Meljanac, and Marijan Mileković. Mappings of multimode bose algebra preserving number operators. *arXiv: High Energy Physics - Theory*, 1994. URL: <https://api.semanticscholar.org/CorpusID:16552148>. (pg. 12, 19)
- [FB70] Uriel Frisch and R Bourret. Parastochastics. *Journal of Mathematical Physics*, 11(2):364–390, 1970. (pg. 12)
- [Fei02] Uriel Feige. Relations between average case complexity and approximation complexity. In *Proceedings of the thirty-fourth annual ACM symposium on Theory of computing*, pages 534–543, 2002. (pg. 2)
- [Fil16] Yuval Filmus. An orthogonal basis for functions over a slice of the boolean hypercube. *The Electronic Journal of Combinatorics*, 23(1), February 2016. URL: <http://dx.doi.org/10.37236/4567>, doi:10.37236/4567. (pg. 10, 15)
- [FTW19] Renjie Feng, Gang Tian, and Dongyi Wei. Spectrum of syk model. *Peking Mathematical Journal*, 2(1):41–70, 2019. (pg. 1, 2, 3)
- [FTW20] Renjie Feng, Gang Tian, and Dongyi Wei. Spectrum of syk model iii: Large deviations and concentration of measures. *Random Matrices: Theory and Applications*, 9(02):2050001, 2020. (pg. 2, 30)
- [FTW21] Renjie Feng, Gang Tian, and Dongyi Wei. Spectrum of syk model ii: Central limit theorem. *Random Matrices: Theory and Applications*, 10(04):2150037, 2021. (pg. 2)
- [FW70] JB French and SSM Wong. Validity of random matrix theories for many-particle systems. *Physics Letters B*, 33(7):449–452, 1970. (pg. 2)
- [GGJV18] Antonio M García-García, Yiyang Jia, and Jacobus JM Verbaarschot. Exact moments of the sachdev-ye-kitaev model up to order $1/n^2$. *Journal of High Energy Physics*, 2018(4):1–43, 2018. (pg. 1, 2, 11, 12)
- [GGV16] Antonio M. García-García and Jacobus J. M. Verbaarschot. Spectral and thermodynamic properties of the sachdev-ye-kitaev model. *Phys. Rev. D*, 94:126010, Dec 2016. URL: <https://link.aps.org/doi/10.1103/PhysRevD.94.126010>, doi:10.1103/PhysRevD.94.126010. (pg. 11, 12)
- [GGV17] Antonio M. García-García and Jacobus J. M. Verbaarschot. Analytical spectral density of the sachdev-ye-kitaev model at finite n . *Phys. Rev. D*, 96:066012, Sep 2017. URL: <https://link.aps.org/doi/10.1103/PhysRevD.96.066012>, doi:10.1103/PhysRevD.96.066012. (pg. 11, 12)
- [GHKM23] Venkatesan Guruswami, Jun-Ting Hsieh, Pravesh K. Kothari, and Peter Manohar. Efficient algorithms for semirandom planted csps at the refutation threshold. In *64th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2023, Santa Cruz, CA, USA, November 6-9, 2023*, pages 307–327. IEEE, 2023. (pg. 10)
- [GKM22] Venkatesan Guruswami, Pravesh K. Kothari, and Peter Manohar. Algorithms and certificates for boolean csp refutation: smoothed is no harder than random. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2022*, page 678–689, New York, NY, USA, 2022. Association for Computing Machinery. doi:10.1145/3519935.3519955. (pg. 2, 10)
- [GM15] Christopher Godsil and Karen Meagher. *Erdos–Ko–Rado theorems: algebraic approaches*, volume 149. Cambridge University Press, 2015. (pg. 10, 15)
- [God10] Chris Godsil. Association schemes, 2010. Lecture notes. URL: <https://www.math.uwaterloo.ca/~cgodsil/pdfs/assoc2.pdf>. (pg. 7, 10)

- [GPSZ26] David Gamarnik, Francisco Pernice, Alexander Schmidhuber, and Alexander Zlokapa. The free energy limit of the syk model at high temperature. *arXiv preprint arXiv:2605.02768*, 2026. (pg. 8)
- [Gro75] Leonard Gross. Logarithmic sobolev inequalities. *American Journal of Mathematics*, 97(4):1061–1083, 1975. URL: <http://www.jstor.org/stable/2373688>. (pg. 18)
- [He26] Yukun He. The spectral edge of the quartic syk model, 2026. URL: <https://arxiv.org/abs/2607.18998>, [arXiv:2607.18998](https://arxiv.org/abs/2607.18998). (pg. 3)
- [HKM23] Jun-Ting Hsieh, Pravesh K. Kothari, and Sidhanth Mohanty. *A simple and sharper proof of the hypergraph Moore bound*, pages 2324–2344. 2023. URL: <https://epubs.siam.org/doi/abs/10.1137/1.9781611977554.ch89>, [arXiv:https://epubs.siam.org/doi/pdf/10.1137/1.9781611977554.ch89](https://epubs.siam.org/doi/pdf/10.1137/1.9781611977554.ch89), [doi:10.1137/1.9781611977554.ch89](https://doi.org/10.1137/1.9781611977554.ch89). (pg. 2, 10)
- [HO22] Matthew B Hastings and Ryan O’Donnell. Optimizing strongly interacting fermionic hamiltonians. In *Proceedings of the 54th annual ACM SIGACT symposium on theory of computing*, pages 776–789, 2022. (pg. 2, 3, 5, 7)
- [HSHT23] Yaroslav Herasymenko, Maarten Stroeks, Jonas Helsen, and Barbara Terhal. Optimizing sparse fermionic hamiltonians. *Quantum*, 7:1081, 2023. (pg. 3)
- [HTS21] Arijit Halder, Omid Tavakol, and Thomas Scaffidi. Variational wave functions for sachdev-ye-kitaev models. *Phys. Rev. Res.*, 3:023020, Apr 2021. URL: <https://link.aps.org/doi/10.1103/PhysRevResearch.3.023020>, [doi:10.1103/PhysRevResearch.3.023020](https://doi.org/10.1103/PhysRevResearch.3.023020). (pg. 2)
- [Iss18] L. Isserlis. On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables. *Biometrika*, 12(1-2):134–139, 11 1918. [arXiv:https://academic.oup.com/biomet/article-pdf/12/1-2/134/481266/12-1-2-134.pdf](https://academic.oup.com/biomet/article-pdf/12/1-2/134/481266/12-1-2-134.pdf), [doi:10.1093/biomet/12.1-2.134](https://doi.org/10.1093/biomet/12.1-2.134). (pg. 16)
- [Itô51] Kiyosi Itô. Multiple Wiener Integral. *Journal of the Mathematical Society of Japan*, 3(1):157 – 169, 1951. [doi:10.2969/jmsj/00310157](https://doi.org/10.2969/jmsj/00310157). (pg. 29)
- [Jan97] Svante Janson. *Gaussian hilbert spaces*. Number 129. Cambridge university press, 1997. (pg. 18, 29)
- [JV18] Yiyang Jia and Jacobus JM Verbaarschot. Large n expansion of the moments and free energy of sachdev-ye-kitaev model, and the enumeration of intersection graphs. *Journal of High Energy Physics*, 2018(11):1–34, 2018. (pg. 11, 12)
- [JZ15] Marius Junge and Qiang Zeng. Ultraproduct methods for mixed q -gaussian algebras. *arXiv preprint arXiv:1505.07852*, 2015. (pg. 12)
- [JZL⁺22] Daniel Jafferis, Alexander Zlokapa, Joseph D Lykken, David K Kolchmeyer, Samantha I Davis, Nikolai Lauk, Hartmut Neven, and Maria Spiropulu. Traversable wormhole dynamics on a quantum processor. *Nature*, 612(7938):51–55, 2022. (pg. 3)
- [Kit15] Alexei Kitaev. A simple model of quantum holography. *Entanglement in strongly-correlated quantum matter*, page 38, 2015. (pg. 1, 2)
- [Kot26] Pravesh K. Kothari. Kikuchi graphs of random hypergraphs are approximately johnson, 2026. URL: <https://arxiv.org/abs/2606.08597>, [arXiv:2606.08597](https://arxiv.org/abs/2606.08597). (pg. 10)

- [Lin22] Henry W Lin. The bulk hilbert space of double scaled syk. *Journal of High Energy Physics*, 2022(11):60, 2022. (pg. 11)
- [LS23] Henry W Lin and Douglas Stanford. A symmetry algebra in double-scaled syk. *SciPost Physics*, 15(6):234, 2023. (pg. 11)
- [LS26] Weihua Liu and Haoqi Shen. Limit joint distributions of syk models with partial interactions, mixed q-gaussian models and asymptotic varepsilon-freeness. *arXiv preprint arXiv:2602.00789*, 2026. (pg. 12)
- [MM96] S. Meljanac and M. Mileković. A unified view of multimode algebras with Fock-like representations. *Int. J. Mod. Phys. A*, 11:1391–1412, 1996. [arXiv:q-alg/9510005](#), [doi:10.1142/S0217751X9600064X](#). (pg. 12, 19)
- [MMP94] S. Meljanac, M. Mileković, and A. Perica. On the r-matrix formulation of deformed algebras and generalized jordan-wigner transformations. *EPL (Europhysics Letters)*, 28:79 – 83, 1994. URL: <https://api.semanticscholar.org/CorpusID:250734085>. (pg. 12, 19)
- [MS16] Juan Maldacena and Douglas Stanford. Remarks on the sachdev-ye-kitaev model. *Phys. Rev. D*, 94:106002, Nov 2016. URL: <https://link.aps.org/doi/10.1103/PhysRevD.94.106002>, [doi:10.1103/PhysRevD.94.106002](#). (pg. 1, 2, 3)
- [MSS16] Juan Maldacena, Stephen H Shenker, and Douglas Stanford. A bound on chaos. *Journal of High Energy Physics*, 2016(8):106, 2016. (pg. 2)
- [Nel66] E. Nelson. A quartic interaction in two dimensions. *Mathematical Theory of Elementary Particles, Proc. Conf., Dedham, Mass., 1965*, pages 69–73, 1966. URL: <https://cir.nii.ac.jp/crid/1570291224381336192>. (pg. 18)
- [Nel73] Edward Nelson. Construction of quantum fields from markoff fields. *Journal of Functional Analysis*, 12(1):97–112, 1973. URL: <https://www.sciencedirect.com/science/article/pii/0022123673900918>, [doi:10.1016/0022-1236\(73\)90091-8](#). (pg. 18)
- [O'D21] Ryan O'Donnell. Analysis of boolean functions, 2021. URL: <https://arxiv.org/abs/2105.10386>, [arXiv:2105.10386](#). (pg. 17, 18, 19, 30)
- [PR16] Joseph Polchinski and Vladimir Rosenhaus. The spectrum in the sachdev-ye-kitaev model. *Journal of High Energy Physics*, 2016(4):1–25, 2016. (pg. 2)
- [PS22] Miguel Pluma and Roland Speicher. A dynamical version of the syk model and the q-brownian motion. *Random Matrices: Theory and Applications*, 11(03):2250031, 2022. (pg. 12)
- [Rio75] John Riordan. The distribution of crossings of chords joining pairs of $2n$ points on a circle. *Mathematics of Computation*, 29(129):215–222, 1975. (pg. 8)
- [Ros19] Vladimir Rosenhaus. An introduction to the syk model. *Journal of Physics A: Mathematical and Theoretical*, 52(32):323001, 2019. (pg. 2)
- [Sac24] Subir Sachdev. Quantum spin glasses and Sachdev-Ye-Kitaev models. 2 2024. [arXiv:2402.17824](#). (pg. 1)
- [Seg56] Irving Ezra Segal. Tensor algebras over hilbert spaces. i. *Transactions of the American Mathematical Society*, 81:106–134, 1956. URL: <https://api.semanticscholar.org/CorpusID:120553155>. (pg. 29)

- [Sim74] Barry Simon. *P(0)2 Euclidean (Quantum) Field Theory*. Princeton University Press, 1974. URL: <http://www.jstor.org/stable/j.ctt13x16st>. (pg. 29)
- [Spe93] Roland Speicher. Generalized statistics of macroscopic fields. *letters in mathematical physics*, 27(2):97–104, 1993. (pg. 12)
- [SY93] Subir Sachdev and Jinwu Ye. Gapless spin-fluid ground state in a random quantum heisenberg magnet. *Phys. Rev. Lett.*, 70:3339–3342, May 1993. URL: <https://link.aps.org/doi/10.1103/PhysRevLett.70.3339>, doi:10.1103/PhysRevLett.70.3339. (pg. 1, 2)
- [Tou52] Jacques Touchard. Sur un probleme de configurations et sur les fractions continues. *Canadian Journal of Mathematics*, 4:2–25, 1952. (pg. 8)
- [Tro15] Joel A. Tropp. An introduction to matrix concentration inequalities. *Foundations and Trends® in Machine Learning*, 8(1-2):1–230, 2015. doi:10.1561/22000000048. (pg. 15)
- [Ver18] Roman Vershynin. *High-dimensional probability: An introduction with applications in data science*, volume 47. Cambridge university press, 2018. (pg. 17)
- [WEAM25] Alexander Wein, Ahmed El Alaoui, and Cristopher Moore. The kikuchi hierarchy and tensor pca. 72(5), 2025. doi:10.1145/3762806. (pg. 2, 10)
- [Wic50] G. C. Wick. The evaluation of the collision matrix. *Phys. Rev.*, 80:268–272, Oct 1950. URL: <https://link.aps.org/doi/10.1103/PhysRev.80.268>, doi:10.1103/PhysRev.80.268. (pg. 16)
- [Wie38] Norbert Wiener. The homogeneous chaos. *American Journal of Mathematics*, 60(4):897–936, 1938. URL: <http://www.jstor.org/stable/2371268>. (pg. 29)
- [Wil90] Richard M. Wilson. A diagonal form for the incidence matrices of t-subsets vs.k-subsets. *European Journal of Combinatorics*, 11(6):609–615, 1990. URL: <https://www.sciencedirect.com/science/article/pii/S0195669813800467>, doi:10.1016/S0195-6698(13)80046-7. (pg. 15)
- [XSS20] Shenglong Xu, Leonard Susskind, Yuan Su, and Brian Swingle. A sparse model of quantum holography. *arXiv preprint arXiv:2008.02303*, 2020. (pg. 3)